Independence for Partition Regular Equations

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Abstract

A matrix A is said to be *partition regular (PR)* over a subset S of the positive integers if whenever S is finitely coloured, there exists a vector x, with all elements in the same colour class in S, which satisfies $Ax = 0$. We also say that *S is PR for A*. Many of the classical theorems of Ramsey Theory, such as van der Waerden's Theorem and Schur's Theorem, may naturally be interpreted as statements about partition regularity. Those matrices which are partition regular over the positive integers were completely characterized by Rado in 1933.

Given matrices A and B, we say that A *Rado-dominates* B if any set which is PR for A is also PR for B. One trivial way for this to happen is if every solution to $Ax = 0$ actually contains a solution to $By = 0$. Bergelson, Hindman and Leader conjectured that this is the only way in which one matrix can Rado-dominate another. In this paper, we prove this conjecture for the first interesting case, namely for 1×3 matrices. We also show that, surprisingly, the conjecture is not true in general.

1 Introduction

A matrix A is said to be *partition regular (PR)* over a subset S of the positive integers if whenever S is finitely coloured, there exists a vector x, with all elements in the same colour class in S, which satisfies $Ax = 0$. We also say that *S is PR for A*, or that 'the system of linear equations $Ax = 0$' is PR over S. All matrices considered in this paper will be finite.

Many of the classical theorems of Ramsey Theory, such as van der Waerden's Theorem [9] and Schur's Theorem [8], may naturally be interpreted as statements about partition regularity. For example, Schur's Theorem states that whenever the positive integers are finitely coloured, one can find x, y and z, all the same colour, with $x + y = z$. This is precisely the statement that the matrix $(1\ 1\ {-1})$ is PR over the set $\mathbb{N}_+$ of positive integers.

When a matrix A is partition regular over $\mathbb{N}_+$, we simply say that the matrix A, or 'the system of linear equations $Ax = 0$', is partition regular. Those matrices which are partition regular (over $\mathbb{N}_+$) were completely characterized by Rado in 1933 [7].

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Given matrices A and B, we say that A *Rado-dominates* B if any set which is PR for A is also PR for B. One trivial way for this to happen is if every solution to $Ax = 0$ actually contains a solution to $By = 0$; when this happens we say that *A solution-dominates B*. Bergelson, Hindman and Leader [1] conjectured that this is the *only* way in which one matrix can Rado-dominate another:

Conjecture 1 ([1]). *Let A and B be partition regular matrices. Then A Rado-dominates B if and only if A solution-dominates B.*

The first non-trivial case of this conjecture is that of 1×3 matrices. Recall that Rado's characterization of PR matrices in the $1 \times n$ case states that the matrix $A = (a_1\, a_2\, \ldots\, a_n)$ is PR if and only if there is some non-empty subset $I \subset [n]$ with $\sum_{i \in I} a_i = 0$, with not all a_i $(i \in I)$ zero. So in the 1×3 case, all PR equations take the form $ax + by - (a+b)z = 0$ or $ax + by - az = 0$ for some positive integers a and b.

The first of these two cases is uninteresting as it has the trivial solution $x = y = z$, and so is solution-dominated by any PR matrix. So we need only consider the second case. Dividing through by a, this becomes $x + \frac{b}{a}y - z = 0$. In other words, the only non-trivial PR equations in three variables are of the form $x + \lambda y = z$ for $\lambda > 0$ rational. So in this case, the conjecture amounts to the statement that, for any distinct positive rationals λ and μ, there is some set $S \subset \mathbb{N}_+$ which is partition regular for the equation $x + \lambda y = z$ but not for the equation $x + \mu y = z$.

We shall prove that Conjecture 1 is true in this case by constructing a set $S \subset \mathbb{N}_+$ which is partition regular for the equation $x + \lambda y = z$, but which contains no solutions *at all* to the equation $x + \mu y = z$. To achieve this aim, it will be enough to construct, for each positive integer k, a finite subset $S_k \subset \mathbb{N}_+$ containing no solutions to $x + \mu y = z$ such that whenever S_k is k-coloured, it contains a monochromatic solution to $x + \lambda y = z$. We may then 'put copies of these sets together' to make S, i.e. we take $S = \bigcup_{k=1}^{\infty} c_k S_k$, where c_1, c_2, c_3, ..., are positive integers chosen, in turn, sufficiently large so as to create no solutions to the equation $x + \mu y = z$ in S. Hence to establish the 1×3 case of Conjecture 1, it is enough to prove that, given any distinct positive rationals λ and μ, and given any positive integer k, there exists a finite subset $S \subset \mathbb{N}_+$ such that, whenever S is k-coloured there exist positive integers a, x, $a + \lambda x \in S$ all the same colour, but such that there are no positive integers b, $y \in S$ with also $b + \mu y \in S$.

The case $\mu \neq 1$ was settled by Bergelson, Hindman and Leader in [1], where they produce a 'fairly' sparse subset S of $\mathbb{N}_+$, which we refer to as a *tree-product* (to be defined later), which is PR for the system $x + \lambda y = z$. In fact, whenever $\mu \neq 1$, this set S can be constructed in such a way that it contains no solutions to $x + \mu y = z$ at all. Unfortunately, it turns out to be rich in solutions to $x + y = z$. This may seem like a mild technical difficulty. However, all known methods of attack on the problem are based on the Hales-Jewett theorem, and create a set S with many solutions to $x + y = z$. So it seems that there is, in fact, a genuine obstacle here. Indeed, this case of the problem has remained open for almost ten years. Our aim in §2 of this paper is to prove Conjecture 1 in this case by establishing the following result:

Theorem 2. *Let $\lambda \neq 1$ be a positive rational and let k be a positive integer. Then there exists a finite subset $S \subset \mathbb{N}_+$ such that whenever S is k-coloured,* *there are positive integers a, b, $a + \lambda b \in S$ all the same colour, but such that S contains no solution to the equation $x + y = z$.*

The main idea of the proof is to replace the use of the Hales-Jewett theorem in the construction by iterated application of some appropriate 'sparse' Hales-Jewett theorem, which will enable us to eliminate the unwanted solutions to $x + y = z$. As we explain below, a reasonable first idea might be to prove an 'intersecting-families' Hales-Jewett theorem. Unfortunately, this result turns out to be false. Our next hope would be an 'antichain' Hales-Jewett theorem, which we are able to prove. This is proved in §3. Our proof of this antichain Hales-Jewett theorem makes use of some extensions of the idea of Nešetřil-Rödl amalgamation.

Rather surprisingly, it turns out that Conjecture 1 is false in general. In §4, we present a counterexample and make some concluding remarks.

We generally use standard notation throughout the paper. We denote by $\mathbb{N}$ the set $\{0, 1, 2, \dots\}$ of natural numbers, and by $\mathbb{N}_+$ the set $\mathbb{N} - \{0\} = \{1, 2, 3, \dots\}$ of positive integers. For $n \in \mathbb{N}_+$, we write $[n]$ to denote the finite set $\{1, 2, \dots n\}$.

2 Avoiding $x + y = z$

Our main aim in this section is to present a proof of Theorem 2. We make extensive use of the Hales-Jewett theorem, of which we shall remind the reader after some necessary definitions. This theorem can be thought of as an 'abstract' version of van der Waerden's theorem.

Let A be a finite set and d a positive integer. We work in A^d, the *d-dimensional Hales-Jewett cube on alphabet A*. A *combinatorial line* in the cube A^d is a set L of the form

$$L = \{(x_1, x_2, \dots, x_d) \in A^d : x_i = x_j \text{ for } i, j \in I,\ x_i = c_i \text{ for } i \in [d] - I\}$$

where I is a non-empty subset of $[d]$ and the c_i ($i \in [d] - I$) are elements of the alphabet A. We call I the set of *active coordinates* of L and $[d] - I$ the set of *inactive coordinates* of L. We are now ready to state the Hales-Jewett theorem.

Theorem 3 (The Hales-Jewett theorem [4]). *Let A be a finite set and k a positive integer. Then there exists a positive integer d such that whenever A^d is k-coloured, it contains a monochromatic line.*

Observe that van der Waerden's theorem follows easily from the Hales-Jewett theorem. It is possible to map the Hales-Jewett cube $[n]^d$ into the positive integers $\mathbb{N}_+$ in such a way that each combinatorial line in $[n]^d$ is taken to an arithmetic progression of length n in $\mathbb{N}_+$: for example, define $\phi\colon [n]^d \to \mathbb{N}_+$ by

$$\phi(x_1, x_2, \dots, x_d) = x_1 + x_2 + \dots + x_d.$$

Then a k-colouring of $\mathbb{N}_+$ induces a k-colouring of $[n]^d$, which, assuming d is sufficiently large, gives a monochromatic line in $[n]^d$, which in turn gives a monochromatic arithmetic progression of length n in $\mathbb{N}_+$.

We shall also require a multi-dimensional extension of this theorem. An *m-dimensional combinatorial subspace* of a Hales-Jewett cube A^d (m = 1, 2, 3, ...) is a set L of the form

$$L = \left\{ (x_1, x_2, \ldots, x_d) \in A^d \ : \ \begin{array}{l} \text{for each } j = 1, 2, \ldots, m, \ x_i = x_h \text{ for } i, h \in I_j, \\ x_i = c_i \text{ for } i \in [d] - \bigcup_{j=1}^m I_j \end{array} \right\}$$

where I_1, I_2, ..., I_m are disjoint non-empty subsets of $[d]$ and the c_i ($i \in [d] - \bigcup_{j=1}^m I_j$) are elements of the alphabet A. We call I_1, I_2, ..., I_m the *active coordinate sets* of L and $[d] - \bigcup_{j=1}^m I_j$ the set of *inactive coordinates* of L. Note that a 1-dimensional combinatorial subspace is simply a combinatorial line.

Theorem 4 (The multi-dimensional Hales-Jewett theorem [4]). *Let A be a finite set, and let k and m be positive integers. Then there exists a positive integer d such that whenever A^d is k-coloured, it contains a monochromatic m-dimensional subspace.*

While this result does not appear explicitly in [4], it follows immediately from Theorem 1 by applying it to the alphabet A^m (see, for example, [3]).

We begin with the aforementioned result of Bergelson, Hindman and Leader. We give a proof in our language as it will be necessary to modify this proof to deal with the case $\mu = 1$. Thus we urge the reader not to skip the proof of Theorem 5.

Before stating the theorem, we need some notation. In what follows, we take λ to be a fixed positive rational.

A *λ-line* is a set of the form $\{x, y, x + \lambda y\}$, where x and y are non-zero elements of $\mathbb{N}$ (or, more generally, of $\mathbb{N}^e$ for some $e \geq 1$).

A *λ-tree of height 0* is a set of the form $\{y\}$ for some non-zero $y \in \mathbb{N}$. A *λ-tree of height 1* is a set of the form $T = \{x, y, x + \lambda y\}$ for non-zero x, $y \in \mathbb{N}$ with $x \neq y$. We say that the λ-tree $\{y\}$ of height 0 is a *pre-tree* of T. A *λ-tree of height h* ($h \geq 2$) is a set T of the form

$$T = R \cup \bigcup_{y \in S - R} \{x_y, y, x_y + \lambda y\}$$

where S is a λ-tree of height $h-1$ with pre-tree R and the x_y ($y \in S - R$) are non-zero elements of $\mathbb{N}$ chosen so that all the x_y and all the $x_y + \lambda y$ ($y \in S - R$) are distinct and not contained in S and so that no unnecessary λ-lines are created in T: in other words, the only λ-lines in T are those in S together with those of the form $\{x_y, y, x_y + \lambda y\}$ for $y \in S - R$. We say that S is a *pre-tree* of T. Note that it is possible to find a λ-tree of any given height h: first construct such a tree in $\mathbb{Q}_+$ by selecting each x_y, in turn, sufficiently large, then multiply every element by some appropriate constant to bring the tree into $\mathbb{N}$.

If T is a tree of height h, we say that $(T_0, T_1, \ldots, T_h)$ is a *tree-sequence* for T if T_i is a tree of height i ($0 \leq i \leq h$), T_i is a pre-tree of T_{i+1} ($0 \leq i \leq h-1$), and $T_h = T$. (The reader may check that, except in the case $\lambda = 1$, the tree-sequence for a given tree is unique.)

A *λ-tree-product of dimension d* is a set $F \subset \mathbb{N}^d$ of the form

$$F = \{(t_1, t_2, \ldots, t_d) : t_i \in T_i \cup \{0\}, t_i \text{ not all } 0\}$$

where T_1, T_2, ..., T_d are λ-trees. We say that F is the *tree-product* of the trees T_1, T_2, ..., T_d. A λ-tree-product is said to be *of height h* if each λ-tree in the definition is of height h.

We are now ready to give the construction of the set required for the 1×3 case of Conjecture 1 in the case $\mu \neq 1$. The proof of the following theorem is simply a rephrasing of the proof from [1]. (While this result does not appear explicitly in [1], it may be read out of Theorem 2.5 of [1] using Lemma 2.6 and Theorem 2.7 of [1].)

Theorem 5 ([1]). *Let λ be a positive rational. Then there exists some λ-tree-product F such that whenever F is k-coloured it contains a monochromatic λ-line.*

Proof. For the remainder of this proof, 'line', 'tree' and 'tree-product' will mean 'λ-line', 'λ-tree' and 'λ-tree-product' respectively.

Let T be a tree of height $k+1$ with tree-sequence $(T_0, T_1, \ldots, T_{k+1})$. Define a finite sequence d_0, d_1, d_2, …, d_{k+1} of positive integers inductively as follows:

- $d_0 = 1$;
- for $1 \leq n \leq k+1$, take d_n sufficiently large that whenever $T_n^{d_n}$ is k-coloured, there exists a monochromatic combinatorial subspace of dimension d_{n-1}.

Note that d_n is guaranteed to exist by the Hales-Jewett theorem.

Now take F to be the tree product of d_{k+1} copies of T.

Suppose F is k-coloured. This induces a k-colouring of the subset $T_{k+1}^{d_{k+1}}$ and so, by our choice of d_{k+1}, we may find a monochromatic d_k-dimensional subspace G_k. We may assume without loss of generality that the active coordinates of G_k are $[d_k]$, i.e. that there exist $z_{d_k+1}, z_{d_k+2}, \ldots, z_{d_{k+1}} \in T_{k+1}$ such that

$$G_k = \{(t_1, t_2, \ldots, t_{d_k}, z_{d_k+1}, z_{d_k+2}, \ldots, z_{d_{k+1}}) : t_1, t_2, \ldots, t_{d_k} \in T_{k+1}\}.$$

[The conscientious reader may be concerned at this point that some of the active coordinate sets of G_k may contain two or more coordinates varying together. But this does not cause a problem—we may simply identify such coordinates by a suitable isomorphism. This will result in a smaller number of inactive coordinates in the set G_k, but the number of inactive coordinates has no bearing on the remainder of the proof. It will, however, be necessary to take more care over keeping track of the coordinates when we come to deal with the case $\mu = 1$.]

Now, write

$$F_k = \{(t_1, t_2, \ldots, t_{d_k}, \underbrace{0, 0, \ldots, 0}_{d_{k+1}-d_k}) : t_1, t_2, \ldots, t_{d_k} \in T_k \cup \{0\}, t_i \text{ not all } 0\}.$$

Note that we may think of F_k as a tree-product of height k by considering it as the tree product of d_k copies of T_k; i.e. we identify F_k with the set

$$\{(t_1, t_2, \ldots, t_{d_k}) : t_1, t_2, \ldots, t_{d_k} \in T_k \cup \{0\}, t_i \text{ not all } 0\}.$$

Now, our original colouring induces a k-colouring of F_k, which in turn gives a k-colouring of the subset $T_k^{d_k}$. By our choice of d_k, we may find a monochromatic d_{k-1}-dimensional subspace G_{k-1}. We may assume without loss of generality that the active coordinate set of G_{k-1} is $[d_{k-1}]$. So there exist $z_{d_{k-1}+1}, z_{d_{k-1}+2}, \ldots, z_{d_k} \in T_k$ such that

$$G_{k-1} = \{(t_1, t_2, \ldots, t_{d_{k-1}}, z_{d_{k-1}+1}, z_{d_{k-1}+2}, \ldots, z_{d_k}) : t_1, t_2, \ldots, t_{d_{k-1}} \in T_k\}.$$

Now, write

$$F_{k-1} = \{(t_1, t_2, \ldots, t_{d_{k-1}}, \underbrace{0, 0, \ldots, 0}_{d_k - d_{k-1}}) : t_1, t_2, \ldots, t_{d_{k-1}} \in T_{k-1} \cup \{0\}, t_i \text{ not all } 0\}.$$

Note that we may think of F_{k-1} as a tree-product of height $k-1$ by considering it as the tree product of d_{k-1} copies of T_{k-1}; i.e. we identify F_{k-1} with the set

$$\{(t_1, t_2, \ldots, t_{d_{k-1}}) : t_1, t_2, \ldots, t_{d_{k-1}} \in T_{k-1} \cup \{0\}, t_i \text{ not all } 0\}.$$

And so we continue. After $k+1$ applications of Hales-Jewett, we have obtained sequences F_0, F_1, $\ldots$, F_k of subsets of F and z_1, z_2, $\ldots$, $z_{d_{k+1}}$ of elements of T satisfying:

- $F_i = \{t_1, t_2, \ldots, t_{d_i}, \underbrace{0, 0, \ldots, 0}_{d_{k+1} - d_i} : t_1, t_2, \ldots, t_{d_i} \in T_i \cup \{0\}, t_i \text{ not all } 0\};$
- the set

  $$G_i = \{t_1, t_2, \ldots, t_{d_i}, z_{d_i+1}, z_{d_i+2}, \ldots, z_{d_{i+1}}, \underbrace{0, 0, \ldots, 0}_{d_{k+1} - d_{i+1}} : t_1, t_2, \ldots, t_{d_i} \in T_{i+1}\}$$

  is monochromatic, with colour c_i, say;
- $z_i \in T_i$ for $i \le d_i$.

Now, by the pigeohole principle, some two of the sets G_0, G_1, $\ldots$, G_k must have the same colour; say $c_m = c_n$ for some $0 \le m < n \le k$. Choose arbitrarily

$$a = (a_1, a_2, \ldots, a_{d_{m+1}}, \underbrace{0, 0, \ldots, 0}_{d_{k+1} - d_{m+1}}) \in G_m.$$

Note that for each i, $1 \le i \le d_{m+1}$, we have $a_i \in T_{m+1}$ and so there is some x_i such that x_i, $x_i + \lambda a_i \in T_{m+2} \subset T_{n+1}$. So, choosing $x_{d_{m+1}+1}$, $x_{d_{m+1}+2}$, $\ldots$, $x_{d_n} \in T_{n+1}$ arbitrarily, and setting $x_i = z_i$ for $d_n + 1 \le i \le d_{n+1}$, we may take

$$x = (x_1, x_2, \ldots, x_{d_{n+1}} \underbrace{0, 0, \ldots, 0}_{d_{k+1} - d_{n+1}}) \in G_n.$$

We now have $a \in G_m$ and x, $x + \lambda a \in G_n$, and so the line $\{a, x, x + \lambda a\}$ is monochromatic with colour $c_m = c_n$. □

This immediately gives us:

Corollary 6 ([1]). *Let λ and μ be distinct positive rationals with $\mu \neq 1$, and let k be a positive integer. Then there exists a subset $S \subset \mathbb{N}_+$ such that*

- *whenever S is k-coloured, it contains a monochromatic λ-line; and*
- *S contains no μ-line.*

Proof. The construction in the proof of Theorem 5 furnishes us with a λ-tree-product $F \subset \mathbb{N}^D$ such that whenever F is k-coloured, it contains a monochromatic λ-line. This tree-product F is the tree-product of D copies of some λ-tree T of height $k+1$. Furthermore, it is clear from the proof of Theorem 5 that if we replace T by *any* λ-tree of the same height then F will still have this Ramsey property.

When constructing a λ-tree, it is clearly possible to avoid creating any μ-lines: when one is called upon to choose an element, one simply takes it to be sufficiently large. Hence we may assume that T has no μ-lines. Similarly, we may also assume that T contains no set of the form $\{w, \mu w\}$ for $w \in \mathbb{N}$.

Now suppose that F contains a μ-line, say x, y and z with $x + \mu y = z$. It is clear that x and y must have disjoint support, as otherwise we could obtain a μ-line in T from x, y and z by projecting onto a coordinate on which x and y were both non-zero. Furthermore, y and z must have disjoint support, as otherwise we could find a set of the form $\{w, \mu w\}$ in T by projecting onto an appropriate coordinate. But this is impossible, as $x + \mu y = z$.

So F contains no μ-line. It simply remains to embed F linearly into $\mathbb{N}_+$ in such a way that no new μ-lines are created; for example, it suffices to take

$$S = \{c_1 t_1 + c_2 t_2 + \cdots + c_D t_D : (t_1, t_2, \ldots, t_D) \in F\},$$

where c_1, c_2, ..., c_D are positive integers selected, in turn, sufficiently large. □

We now turn to the case $\mu = 1$.

Given a positive rational $\lambda \neq 1$, it is of course possible to construct λ-trees of arbitrary height containing no x, y and z with $x + y = z$. Unfortunately, when we then take tree-products of such structures, the resulting λ-tree-product F is rich in solutions to $x + y = z$: indeed, whenever x, $y \in F$ have disjoint support then also $x + y \in F$.

However, this is the only way in which things can go wrong. In other words, whenever we have a solution to $x + y = z$ in F then x and y have disjoint support. So it will suffice to exclude such triples from our structure.

Our method begins by taking a large tree-product F. We then try to construct a subset S of F which is sufficiently sparse that it does not contain x, y and z such that x and y have disjoint support whose union is the support of z, but which is sufficiently rich in structure that the proof of Theorem 5 can still be pushed through.

How can this be achieved? The proof of Theorem 5 begins by applying the Hales-Jewett theorem to $T^{d_{k+1}}$ to extract a d_k-dimensional subspace G_k, so we will need S to contain the whole of $T^{d_{k+1}}$. So far, so good—every point of $T^{d_{k+1}}$ is supported on the whole of $[d_{k+1}]$.

After extracting our monochromatic subspace G_k, we need to consider points of F whose support is the union of the active coordinate sets of G_k. This is where the trouble starts: we can easily find three possible candidates for our monochromatic G_k, say $G_k^{(1)}$, $G_k^{(2)}$ and $G_k^{(3)}$, whose active coordinate sets have unions I_1, I_2 and I_3 respectively, such that I_1 and I_2 are disjoint with union I_3. Then for the proof to go through as before, we need to consider points supported on each of I_1, I_2 and I_3, and the construction has failed. A similar problem arises if we have I_1 and I_2 disjoint with union the whole of $[d_{k+1}]$.

We come here to the key idea in our construction: what if we were somehow able to restrict the possible candidates for G_k so that this undesirable situation could not arise? In other words, instead of considering *all* d_k-dimensional subspaces of $T^{d_{k+1}}$, what if we only consider some suitable subcollection $\mathcal{G}$ of subspaces?

For the proof that whenever S is k-coloured it contains a monochromatic solution to $x + \lambda y = z$ to go through, we will need one of the subspaces in $\mathcal{G}$ to be monochromatic whenever $T^{d_{k+1}}$ is k-coloured.

Let $\mathcal{I}$ denote the multi-set of subsets $I_G \subset [d_{k+1}]$ each of which is the union of the active coordinate sets of some $G \in \mathcal{G}$. An obvious way to avoid the problem above would be to insist that $\mathcal{I}$ be an *intersecting family*, i.e. that $I_G \cap I_H \neq \emptyset$ for all G, $H \in \mathcal{G}$. So our first hope would be to prove an 'intersecting-family' Hales-Jewett theorem. In the one-dimensional case (i.e. when we are looking for a monochromatic line), this would say that given positive integers n and k we can find a positive integer d and a collection $\mathcal{L}$ of lines in $[n]^d$ such that whenever $[n]^d$ is k-coloured one of the lines in $\mathcal{L}$ is monochromatic, and such that the active coordinate sets of the lines in $\mathcal{L}$ form an intersecting family. Unfortunately, this is not true. For suppose that we have n and d and a collection $\mathcal{L}$ of lines whose active coordinate sets form an intersecting family. If any two lines L, $L' \in \mathcal{L}$ intersect in a point x, say, and x is the ith point on L then also x is the ith point on L' (as L and L' have an active coordinate in common). So we can 2-colour $[n]^d$ in such a way that all x which are the first point of some line in $\mathcal{L}$ are red, and all x which are the second point of some line in $\mathcal{L}$ are blue.

After this failure, our next hope would be to prove an 'antichain' Hales-Jewett theorem, which at least is not trivially false. It would suffice to have $\mathcal{I}$ an antichain, with the additional two properties that $[d_{k+1}] \notin \mathcal{I}$ and that there are no I_G, $I_H \in \mathcal{I}$ disjoint with $I_G \cup I_H = [d_{k+1}]$. Fortunately, this result does turn out to be true. In the one-dimensional (monochromatic line) case, what we need is:

Theorem 7 (Antichain Hales-Jewett theorem). *Let $n \geq 3$ and k be positive integers. Then there exists some positive integer d and a collection $\mathcal{L}$ of lines in the Hales-Jewett cube $[n]^d$ such that*

- *whenever $[n]^d$ is k-coloured, one of the lines in $\mathcal{L}$ is monochromatic;*
- *the active coordinate sets of distinct lines in $\mathcal{L}$ are incomparable—in other words, if L and L' are distinct lines in $\mathcal{L}$ with active coordinate sets I and I' respectively, then $I \not\subset I'$ and $I' \not\subset I$.*

We defer the proof of Theorem 7 to §3. For now, we continue with our discussion of how to construct the required set S using Theorem 7.

We of course need the further conditions that no line in $\mathcal{L}$ has active coordinate set the whole of $[d]$, and that no two lines in $\mathcal{L}$ have disjoint active coordinate sets whose union is the whole of $[d]$. But this is easy: once we have constructed $[n]^d$ and $\mathcal{L}$ satisfying the conclusions of Theorem 7, we may simply add an extra dimension to our Hales-Jewett cube.

More generally, what we need is a *multidimensional* antichain Hales-Jewett theorem rather than the one-dimensional version stated above. It is a trivial matter to deduce such a result from the one-dimensional version, by exactly the same method as the multidimensional version of the standard Hales-Jewett theorem is deduced from the one-dimensional version.

Corollary 8 (Multidimensional antichain Hales-Jewett theorem). *Let $n \geq 3$, k and m be positive integers. Then there exists some positive integer d (divisible by m) and a family $\mathcal{A} \subset \mathcal{P}[d/m]$ such that*

- *whenever $[n]^d$ is k-coloured, it contains a monochromatic m-dimensional subspace whose active coordinate sets $I_1, I_2, \ldots, I_m$ are of the form*

$$I_j = \{m(i-1) + j : i \in I\}$$

  for some $I \in \mathcal{A}$;

- *$\mathcal{A}$ is an antichain; and*

- *there are no two disjoint sets in $\mathcal{A}$ with union the whole of $[d/m]$, and $[d/m] \notin \mathcal{A}$.*

Proof. Apply Theorem 7 with alphabet $[n]^m$. □

What remains is to check that this procedure can be iterated. In other words: if we apply this antichain Hales-Jewett theorem at each stage of the proof, and take S to consist of only those points of F necessary for the argument that there is a monochromatic solution to $x + \lambda y = z$, then does S remain free of solutions to $x + y = z$?

Fix a positive rational $\lambda \neq 1$. Throughout this construction, and throughout the following proof that the set constructed does indeed have the properties that we claim, we use 'line', 'tree', and 'tree-product' to mean 'λ-line', 'λ-tree' and 'λ-tree-product' respectively.

As before, we begin by constructing a tree T of height $k+1$ with tree-sequence $(T_0, T_1, \ldots, T_{k+1})$, and we can of course do this in such a way that T contains no x, y and z with $x + y = z$.

Next, we inductively construct sequences $d_0, d_1, \ldots, d_{k+1}$ of positive integers (with $d_{i-1} | d_i$ for $i = 1, 2, \ldots, k+1$) and sets $\mathcal{A}_{i-1} \subset \mathcal{P}[d_i/d_{i-1}]$ ($i = 1, 2, \ldots, k+1$) as follows:

- $d_0 = 1$;

- for $1 \leq i \leq k+1$, take d_i sufficiently large (and divisible by d_{i-1}) and $\mathcal{A}_{i-1} \subset \mathcal{P}[d_i/d_{i-1}]$ an antichain containing no two disjoint sets with union the whole of $[d_i/d_{i-1}]$, with $[d_i/d_{i-1}] \notin \mathcal{A}_{i-1}$, such that whenever $T_i^{d_i}$ is k-coloured, it contains a monochromatic d_{i-1}-dimensional subspace with active coordinate sets $I_1, I_2, \ldots, I_{d_{i-1}}$ given by

$$I_j = \{d_{i-1}(l-1) + j : l \in I\}$$

  for some $I \in \mathcal{A}_{i-1}$. (Note that this is of course possible by Corollary 8.)

For convenience, we define $\mathcal{A}_{k+1} = \{\{1\}\}$.

Now, take F to be the tree-product of d_{k+1} copies of T. As this definition is stated, we currently have $F \subset \mathbb{N}^{d_{k+1}}$. However, in a similar manner to the proof of Corollary 6, it is easy to linearly embed F into $\mathbb{N}_+$ in such a way that the only solutions of $x + y = z$ in this embedded copy of F are images of solutions in $F \subset \mathbb{N}^{d_{k+1}}$.

We now proceed to construct $S \subset F$ containing no solutions to $x + y = z$, but still PR for the equation $x + \lambda y = z$.

For each integer j with $0 \leq j \leq k+1$, and for each $A_{k+1} \in \mathcal{A}_{k+1}$, $A_k \in \mathcal{A}_k$, $\ldots$, $A_j \in \mathcal{A}_j$, define $S_{A_{k+1}, A_k, \ldots, A_j}$ to consist of those $x \in F$ which satisfy

- $x_\alpha \neq 0$ precisely when α can be written in the form
$$\alpha = (a_{k+1}-1)d_{k+1} + (a_k-1)d_k + \cdots + (a_j-1)d_j + r$$
with $a_{k+1} \in A_{k+1}$, $a_k \in A_k$, $\ldots$, $a_j \in A_j$ and $1 \leq r \leq d_j$;
- $x_\alpha \in T_j \cup \{0\}$ for all α; and
- x_α depends only on the residue class of α modulo d_j.

We take S to be the union of all the $S_{A_{k+1},A_k,\ldots,A_j}$.

We write $\mathrm{supp}(x)$ for the set of coordinates α with $\alpha \neq 0$. Given j with $0 \leq j \leq k+1$, and given $A_{k+1} \in \mathcal{A}_{k+1}$, $A_k \in \mathcal{A}_k$, $\ldots$, $A_j \in \mathcal{A}_j$, note that $\mathrm{supp}(x)$ is constant for $x \in S_{A_{k+1},A_k,\ldots,A_j}$; hence we may write $\mathrm{supp}(S_{A_{k+1},A_k,\ldots,A_j})$ to mean $\mathrm{supp}(x)$ for $x \in S_{A_{k+1},A_k,\ldots,A_j}$.

Given j with $0 \leq j \leq k+1$, $A_{k+1} \in \mathcal{A}_{k+1}$, $A_k \in \mathcal{A}_k$, $\ldots$, $A_j \in \mathcal{A}_j$ and $f\colon [d_{j+1}] - \bigcup_{a \in A_j} [(a-1)d_j+1, ad_j] \to T_{j+1}$, let $S^f_{A_{k+1},A_k,\ldots,A_j}$ be the collection of points $x \in S_{A_{k+1},A_k,\ldots,A_{j+1}}$ satisfying

- if $\alpha \in \mathrm{supp}(x)$ and $\alpha \equiv r \pmod{d_{j+1}}$ for some $r \in \mathrm{dom}(f)$ then $x_\alpha = f(r)$; and
- if α, $\beta \in \mathrm{supp}(S_{A_{k+1},A_k,\ldots,A_j})$ with $\alpha \equiv \beta \pmod{d_j}$ then $x_\alpha = x_\beta$.

Observe that

- the $S_{A_{k+1},A_k,\ldots,A_j}$ are pairwise disjoint;
- we have $\mathrm{supp}(S_{A_{k+1},A_k,\ldots,A_j}) \subset \mathrm{supp}(S_{A'_{k+1},A'_k,\ldots,A'_{j'}})$ if and only if $(A_{k+1}, A_k, \ldots, A_j)$ is an initial segment of $(A'_{k+1}, A'_k, \ldots, A'_j)$, and the containment is strict when it is a proper initial segment;
- whenever $S_{A_{k+1},A_k,\ldots,A_j}$ ($j \geq 1$) is k-coloured, we can find $A_{j-1} \in \mathcal{A}_{j-1}$ and $f\colon [d_j] - \bigcup_{a \in A_{j-1}} [(a-1)d_{j-1}+1, ad_{j-1}] \to T_j$ such that the set $S^f_{A_{k+1},A_k,\ldots,A_{j-1}}$ is monochromatic.

It remains now to show that this S does indeed have the promised properties.

Theorem 9. *Let $\lambda \neq 1$ be a positive rational, and let k be a positive integer. Then there exists a subset $S \subset \mathbb{N}_+$ such that*

- *whenever S is k-coloured, it contains a monochromatic λ-line; and*
- *S does not contain x, y and z with $x+y=z$.*

Proof. We construct F and S as above. As this definition stands, S is a subset of $\mathbb{N}^{d_{k+1}}$ rather than of $\mathbb{N}$; but this is unimportant because, precisely as in the proof of Corollary 6, we can embed S linearly in $\mathbb{N}_+$ in such a way that no solutions to $x+y=z$ are created.

The proof that a k-colouring of S always yields a monochromatic λ-line is almost identical to the proof of Theorem 5. The only change needed is that wherever the multidimensional Hales-Jewett theorem was applied before, we now replace it with Corollary 8.

To be more precise, suppose that S is k-coloured. By our construction of S using Corollary 8, there exist $A_{k+1} \in \mathcal{A}_{k+1}$, $A_k \in \mathcal{A}_k$, $\ldots$, $A_0 \in \mathcal{A}_0$ and, for

$0 \le j \le k$, a function $f_j \colon [d_{j+1}] - \bigcup_{a \in A_j} [(a-1)d_j + 1, ad_j]$ such that for each j with $0 \le j \le k$ the set

$$G_j = S^{f_j}_{A_{k+1}, A_k, \ldots, A_j}$$

is monochromatic.

Now there must be i and j with $0 \le i < j \le k$ such that G_i and G_j have the same colour; precisely as in the proof of Theorem 5, we find $y \in G_i$ and $x \in G_j$ such that also $x + \lambda y \in G_j$. So we have found a monochromatic λ-line as required.

We now come to the second assertion of our theorem. Assume for a contradiction that there do exist x, y, $z \in S$ with $x + y = z$. Then x and y must have disjoint support as otherwise, by considering a coordinate on which both x and y are non-zero, we find x', y', $z' \in T$ with $x' + y' = z'$. Hence we have $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$ and $\mathrm{supp}(x) \cup \mathrm{supp}(y) = \mathrm{supp}(z)$.

As $\mathrm{supp}(x)$, $\mathrm{supp}(y) \subsetneq \mathrm{supp}(z)$, we must have

$$\begin{aligned} z &\in S_{A_{k+1}, A_k, \ldots, A_j} \\ x &\in S_{A_{k+1}, A_k, \ldots, A_j, A_{j-1}, \ldots, A_i} \\ y &\in S_{A_{k+1}, A_k, \ldots, A_j, A'_{j-1}, \ldots, A'_{i'}} \end{aligned}$$

for some integers i, i', j with $0 \le i, i' < j \le k+1$ and some $A_{k+1} \in \mathcal{A}_{k+1}$, $A_k \in \mathcal{A}_k$, $\ldots$, $A_i \in \mathcal{A}_i$, $A'_{j-1} \in \mathcal{A}_{j-1}$, $A'_{j-2} \in \mathcal{A}_{j-2}$, $\ldots$, $A'_{i'} \in \mathcal{A}_{i'}$.

As $\mathrm{supp}(x) \cup \mathrm{supp}(y) = \mathrm{supp}(z)$, we must have $A_{j-1} \ne A'_{j-1}$; for otherwise

$$\mathrm{supp}(x), \mathrm{supp}(y) \subset \mathrm{supp}(S_{A_{k+1}, A_k, \ldots, A_{j-1}}) \subsetneq \mathrm{supp}(S_{A_{k+1}, A_k, \ldots, A_j}) = \mathrm{supp}(z).$$

Furthermore, $A_{j-1} \cup A'_{j-1} = [d_j / d_{j-1}]$. Hence $A_{j-1} \cap A'_{j-1} \ne \emptyset$.

As $\mathcal{A}_{j-1}$ is an antichain, we may assume without loss of generality that there is some $a_{j-1} \in A_{j-1} - A'_{j-1}$. Choose $a_{k+1} \in A_{k+1}$, $a_k \in A_k$, $\ldots$, $a_j \in A_j$, and for $1 \le r \le d_{j-1}$ consider

$$\alpha_r = (a_{k+1} - 1)d_{k+1} + (a_k - 1)d_k + \cdots + (a_{j-1} - 1)d_{j-1} + r.$$

Then for each r, $\alpha_r \in \mathrm{supp}(z)$ but $\alpha_r \notin \mathrm{supp}(y)$, and so $\alpha_r \in \mathrm{supp}(x)$. But this implies that $x \in S_{A_{k+1}, A_k, \ldots, A_{j-1}}$.

Now choose $a'_{j-1} \in A_{j-1} \cap A'_{j-1}$, and for $1 \le r \le d_{j-1}$ consider

$$\alpha'_r = (a_{k+1} - 1)d_{k+1} + (a_k - 1)d_k + \cdots + (a_j - 1)d_j + (a'_{j-1} - 1)d_{j-1} + r.$$

Then $\alpha'_r \in \mathrm{supp}(x)$ for all r, but also $\alpha'_r \in \mathrm{supp}(y)$ for some r. But this contradicts $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$.

So indeed we cannot have x, y, $z \in S$ with $x + y = z$. This concludes our proof. □

3 Proof of the antichain Hales-Jewett theorem

We now turn to the proof of Theorem 7. The ideas of the proof are based on the notion of 'amalgamation' introduced by Nešetřil and Rödl [5], and developments of this work by Frankl, Graham and Rödl [2]. (See also Prömel and Voigt [6].)

It will be convenient for the proof to insist not only that the active coordinate sets of lines in $\mathcal{L}$ form an antichain, but that the same is true if we consider *all* lines contained within the union of the lines in $\mathcal{L}$. In other words, we shall prove Theorem 7 in the following stronger form:

Theorem 10. *Let $n \geq 3$ and k be positive integers. Then there exists some positive integer d and a subset S of the Hales-Jewett cube $[n]^d$ such that*

- *whenever S is k-coloured it contains a monochromatic line; and*
- *the multi-set $\mathcal{A}(S)$ of active coordinate sets of lines in S is an antichain.*

We note that Theorem 7 follows instantly from Theorem 10.

We begin by fixing some d_0 such that whenever $[n]^{d_0}$ is k-coloured, it contains a monochromatic line. We can of course do this by the ordinary Hales-Jewett theorem. The Hales-Jewett cube $[n]^{d_0}$ will be used to index the sets in the forthcoming amalgamation.

We define a *picture* S inside some Hales-Jewett cube $[n]^d$ to be a collection of pairwise disjoint sets $S_v \subset [n]^d$ $(v \in [n]^{d_0})$. We call the set $\bigcup_{v\in[n]^{d_0}} S_v$ the *ground-set* of S, and often denote it simply by S. For any picture S, we denote the multi-set of active-coordinate-sets of lines in S by $\mathcal{A}(S)$.

Our first task is to construct our starting picture S inside some large $[n]^d$. We consider $[n]^d$ as coming equipped with the *pointwise ordering;* in other words, given $a = (a_1, a_2, \ldots, a_d) \in [n]^d$ and $b = (b_1, b_2, \ldots, b_d) \in [n]^d$, we write $a \leq b$ to mean $a_i \leq b_i$ for all $i = 1, 2, \ldots, d$. This starting picture S must satisfy

- 'for every line in $[n]^{d_0}$ there is a corresponding line in S', i.e. if $\{x_1, x_2, \ldots, x_n\}$ is a line in $[n]^{d_0}$ with $x_1 < x_2 < \cdots < x_n$ then there is a line $\{y_1, y_2, \ldots, y_n\}$ in S with $y_1 < y_2 < \cdots < y_n$ and $y_i \in S_{x_i}$ for each i $(1 \leq i \leq n)$;
- each S_v is an antichain (in the pointwise ordering on $[n]^d$); and
- the multi-set $\mathcal{A}(S)$ of active-coordinate-sets of lines in S is an antichain in $\mathcal{P}[d]$.

Such a picture is easy to find if we take d sufficiently large: for each line in $[n]^{d_0}$, select a line in $[n]^d$, the active coordinate sets of these lines all *disjoint*, and do so in such a way that no two points are related unless they are both in one of the lines we select.

We now come to the amalgamation.

Suppose that we have some picture S, inside a Hales-Jewett cube $[n]^d$, satisfying the final two conditions above, namely that each S_v is an antichain and that $\mathcal{A}(S)$ is an antichain. Suppose also that we are given some fixed $u \in [n]^{d_0}$. We want to find a picture S' inside some large Hales-Jewett cube which also satisfies these two conditions, *and* such that whenever S' is k-coloured, it contains a copy of S with S_u monochromatic. We call S' the *amalgamation of S over S_u*, and define it as follows:

> First fix some e such that, whenever $(S_u)^e$ is k-coloured, it contains a monochromatic S_u-line. (We can obviously do this by applying the Hales-Jewett theorem to the alphabet S_u.) Note that such an S_u-line is a copy of S_u inside $[n]^{de}$. Denote all of the S_u-lines in

$(S_u)^e$ by $S_u^{(1)}$, $S_u^{(2)}$, …, $S_u^{(D)}$. For each $v \in [n]^{d_0}$ and each i with $1 \le i \le D$, let $S_v^{(i)}$ denote the copy of S_v corresponding to $S_u^{(i)}$; in other words, $S_v^{(i)}$ is the image of S_v under that canonical embedding of $[n]^d$ as a d-dimensional subspace of $[n]^{de}$ which maps S_u onto $S_u^{(i)}$.

We now define our goal picture S' inside $\left([n]^{d_0}\right)^D \times [n]^{de}$ by setting

$$S_v'^{(i)} = (\underbrace{u, u, \dots .u}_{i-1}, v, \underbrace{u, u, \dots , u}_{D-i}, S_v^{(i)})$$

for each $v \in [n]^{d_0}$ and $1 \le i \le D$, and defining

$$S_v' = \bigcup_{i=1}^{D} S_v'^{(i)}.$$

Note in particular that

$$S_u' = (\underbrace{u, u, \dots , u}_{D}, (S_u)^e).$$

Note also that for each i with $1 \le i \le D$ we may define a picture $S^{(i)}$ by $(S^{(i)})_v = S_v^{(i)}$, and that each such picture $S^{(i)}$ is a copy of the picture S. It is also easy to see precisely the structure of these copies of S: given $S^{(i)}$, there exists a non-empty set $I \subset [e]$, and constants $c_j \in S_u$ for each $j \in [e] - I$, such that $S_v^{(i)}$ contains precisely those $x \in ([n]^{d_0})^D \times S^e$ for which

- $x_i = v$;
- $x_j = u$ for $1 \le j \le D$ and $j \ne i$;
- $x_{D+j} = c_j$ for $j \in [e] - I$; and
- $x_{D+j} = x_{D+j'} \in S_v$ for j, $j' \in I$.

It remains to show that S' does indeed have the properties that we claim.

Lemma 11. *Let S be a picture inside some $[n]^d$ such that each of the sets S_v is an antichain, and such that the multi-set $\mathcal{A}(S)$ is an antichain. Let S' be the amalgamation of S over S_u for some $u \in [n]^{d_0}$. Then*

(i) whenever S' is k-coloured, it contains a copy of S with S_u monochromatic;

(ii) each of the sets S_v' is an antichain; and

(iii) the multi-set $\mathcal{A}(S')$ is an antichain.

Proof. (i) Suppose that S' is k-coloured. This induces a k-colouring of $(S_u)^e$—we simply restrict the given colouring to S_u', which is a copy of $(S_u)^e$. Now, by choice of e, we can find a monochromatic S_u-line inside $(S_u)^e$, say $S_u^{(i)}$. But then the copy $S^{(i)}$ of S has $S_u^{(i)}$ monochromatic as required.

(ii) There are two cases to consider.

In the case $v \ne u$, each $S_v^{(i)}$ is an antichain by assumption. Moreover, if $x \in S_v^{(i)}$ and $y \in S_v^{(j)}$ for some i, j with $i \ne j$ then x has v and u in coordinates

i and j respectively, while y has u and v in these positions, so x and y are incomparable. Hence $S'_v = \bigcup_{i=1}^{D} S_v^{(i)}$ is itself an antichain.

In the other case, where $v = u$, we know that S_u is an antichain, and so S_u^e is an antichain, and so $S'_u = (\underbrace{u, u, \ldots, u}_{D}, S_u^e)$ is an antichain.

(iii) Finally, we must show that the active coordinate sets of the lines in S' form an antichain. So let L and L' be distinct lines in S', with active coordinate sets J and J' respectively.

We begin by showing that L is contained entirely within a single copy of S. For this purpose, we think of S' as a subset of $([n]^{d_0})^D \times S^e$; i.e. we consider each point of S' to have $D + e$ coordinates, the first D drawn from $[n]^{d_0}$ and the final e drawn from S. Note that, in each coordinate, L must either form a line (in $[n]^{d_0}$ or S as appropriate) or be constant.

Consider the first D coordinates. If $x \in L$ has $v \neq u$ in any of these positions, it must lie in S'_v; but S'_v is an antichain, so for each such v this can happen for at most one $x \in L$. Hence when L is constant, it must be constant with value u. But if $x \in L$ has u in *every* one of the first D positions then it must lie in S'_u; but S'_u is an antichain, so this can happen for at most one $x \in L$. Hence L forms a line in at least one of the first D coordinates. Call this coordinate i. Then any $x \in L$ with $x \notin S'_u$ must lie in $S^{(i)}$. As $n \geq 3$ and L can contain at most one point from S'_u, this means that L contains at least two points of $S^{(i)}$. But it is immediate from the structure of $S^{(i)}$ as described above that any line with two points in $S^{(i)}$ must lie entirely in $S^{(i)}$. So L is entirely contained within the single copy $S^{(i)}$ of S.

Similarly, L' lies entirely inside some copy $S^{(j)}$ of S.

We now return to thinking of S' as a subset of $[n]^{Dd_0+ed}$. Amongst the first D blocks of d_0 coordinates, L has active coordinates within the ith block but in no other, and L' has active coordinates in the jth block but no other. So if $i \neq j$ then J and J' are incomparable.

Finally, assume that $i = j$. Take $I \subset [e]$ as in the description of the structure of $S^{(i)}$ immediately before this lemma. Pick $\beta \in I$, and consider the projections of L and L' onto the block B of coordinates corresponding to the βth copy of S in the Cartesian product $([n]^{d_0})^D \times S^e = [n]^{Dd_0+ed}$, i.e. $B = [d_0 D + (\beta - 1)d + 1, d_0 D + \beta d]$. Each of these two projections is a line in S, and these lines must be distinct, as otherwise L and L' would be identical. But as $\mathcal{A}(S)$ is an antichain, this implies that $J \cap B$ and $J' \cap B$ are incomparable, and so J and J' are incomparable.

Hence $\mathcal{A}(S')$ is an antichain. □

We are now ready to prove our antichain Hales-Jewett theorem.

Proof (of Theorem 10). We first take a starting picture S in $[n]^d$, and some $u \in [n]^{d_0}$, and amalgamate over S_u to form a picture S'. We next choose some $v \in [n]^{d_0}$ with $v \neq u$ and form the amalgamation S'' of S' over S'_v. Then we take some $w \in [n]^{d_0}$ with $w \neq u$, v and form the amalagamation S''' of S'' over S''_w. Continuing in this way, and amalgamating n^{d_0} times in total (once for each element $t \in [n]^{d_0}$), we eventually obtain a picture S^*, say. By the preceding lemma, the ground-set of S^* has the properties we require and the theorem is proved. □

4 A counterexample

We now present a counterexample to Conjecture 1. Specifically, we show that the system $x+y+u=z$ is Rado-dominated by the system $x+y=z$. The main idea of the construction is to 'augment' a colouring in a certain useful way.

Theorem 12. *Let S be a set which is PR for the equation $x+y=z$. Then S is also PR for the equation $x+y+u=z$.*

Proof. Let c be a k-colouring of S. Define a $2k$-colouring c' of S by setting

$$c'(z) = \begin{cases} (c(z),1) & \text{if there exist } x,\, y \in S \text{ of colour } c(z) \text{ with } x+y=z \\ (c(z),0) & \text{otherwise} \end{cases} .$$

As S is PR for the equation $x+y=z$, we can find x, y, $z \in S$ with $c'(x)=c'(y)=c'(z)$ and $x+y=z$. In particular, $c(x)=c(y)=c(z)=c$, say, and so $c'(z)=(c,1)$. But then $c'(x)=(c,1)$ and so there exist u, $v \in S$ with $c(u)=c(v)=c$ and $u+v=x$. But now we have $u+v+y=z$ and $c(u)=c(v)=c(y)=c(z)$ as required. So S is also PR for the equation $x+y+u=z$. □

This still leaves open the question of precisely when one PR matrix Rado-dominates another.

There are many cases where, given PR matrices A and B with B not solution-dominated by A, it is easy to modify the proof of Theorem 2 to show that A does not Rado-dominate B. Indeed, [1] provides an appropriate definition of a tree corresponding to any given PR matrix A, so that k-colouring a large tree-product always leaves a monochromatic solution to $Ax=0$.

In certain cases, some more work is required to allow this approach to succeed. For example, suppose we wish to construct $S \subset \mathbb{N}_+$ which is partition regular for the equation $x+\lambda y=z$ ($\lambda \neq 1$) but not for the equation $x+y+z=w+t$. This is still possible, but more extensive modifications to the proof of Theorem 2 are necessary. In particular, we need a stronger Hales-Jewett-style theorem: specifically, in Theorem 7 we must replace the condition '$\mathcal{A}$ is an antichain' with the stronger 'if I, J, $K \in \mathcal{A}$ with $I \subset J \cup K$ then either $I=J$ or $I=K$'. This stronger result can be proved similarly to Theorem 7.

Unfortunately, there are other cases where it appears that a similar method of proof cannot succeed. For example, suppose that A is the matrix $(1\ 2\ 3\ -1)$ and B is the matrix $(1\ 2\ -1)$. Note that A does not solution-dominate B. However, taking the definition of a tree corresponding to A from [1], one finds that even a single tree will always contain solutions to $By=0$. We do not know whether or not A Rado-dominates B.

In the other direction, one can write down many counterexamples to Conjecture 1, but all the counterexamples we have found are derived from the same basic idea as Theorem 12. We do not know whether or not this phenomenon represents the only 'obstruction' to Conjecture 1.

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