Solutions to Example Sheet 4.

1. We only need to check that if $\mathcal{F} \to \mathcal{F}''$ is surjective, so is $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$.

By the quoted fact, there exist A-modules M, M'' with $\mathcal{F} = \widetilde{M}$, $\mathcal{F}'' = \widetilde{M}''$. In particular, as surjectivity of $\mathcal{F} \to \mathcal{F}''$ can be checked on stalks and $(\widetilde{M})_{\mathfrak{p}} = M_{\mathfrak{p}}$ for any $\mathfrak{p} \in \operatorname{Spec} A$, necessarily $M_{\mathfrak{p}} \to M''_{\mathfrak{p}}$ is surjective for all $\mathfrak{p}$. As a homomorphism of A-modules being surjective is a local propery, $M \to M''$ is surjective. But $M = \Gamma(X, \mathcal{F})$, $M'' = \Gamma(X, \mathcal{F}'')$, giving the desired result.

2. Let $\mathcal{Z}^i = \ker d^i$, so in particular $\mathcal{F} = \mathcal{Z}_0$. Then the given long exact sequence splits up into short exact sequences

$$0 \to \mathcal{Z}^i \to \mathcal{F}^i \to \mathcal{Z}^{i+1} \to 0. \tag{0.1}$$

Taking the long exact sequence of cohomology and using the assumed vanishing, we get an exact sequence

$$0 \to H^0(X, \mathcal{Z}^i) \to H^0(X, \mathcal{F}^i) \to H^0(X, \mathcal{Z}^{i+1}) \to H^1(X, \mathcal{Z}^i) \to 0$$

and

$$H^p(X, \mathcal{Z}^{i+1}) \cong H^{p+1}(X, \mathcal{Z}^i) \tag{0.2}$$

for $p \geq 1$. Thus

$$H^0(X, \mathcal{F}) = H^0(X, \mathcal{Z}^0) = \ker(d^0 : H^0(X, \mathcal{F}^0) \to H^0(X, \mathcal{Z}^1)).$$

However, as $\mathcal{Z}^1$ injects into $\mathcal{F}^1$, $H^0(X, \mathcal{Z}^1)$ injects into $H^0(X, \mathcal{F}^1)$, and we can write

$$H^0(X, \mathcal{F}) = \ker(d^0 : H^0(X, \mathcal{F}^0) \to H^0(X, \mathcal{F}^1)),$$

as claimed.

Next, for $p > 0$, we have

$$H^p(X, \mathcal{F}) = H^p(X, \mathcal{Z}^0) \cong H^{p-1}(X, \mathcal{Z}^1) \cong \cdots \cong H^1(X, \mathcal{Z}^{p-1}),$$

by repeated use of (0.2). Now $H^0(X, \mathcal{Z}^p) = \ker d^p : H^0(X, \mathcal{F}^p) \to H^0(X, \mathcal{F}^{p+1})$, and by (0.1), $H^1(X, \mathcal{Z}^{p-1}) = \operatorname{coker} d^{p-1} : H^0(X, \mathcal{F}^{p-1}) \to H^0(X, \mathcal{Z}^p)$. This gives the desired form for $H^p(X, \mathcal{F})$.

3. (a) Let $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$ and $\beta : \mathcal{F}_2 \to \mathcal{F}_3$ be the two given maps. We begin by observing how flabbyness is used here. Suppose given open sets U_1, U_2 of X and sections $t_i \in \Gamma(U_i, \mathcal{F}_2)$ with the property that $\beta(t_1) = \beta(t_2)$ on $U_1 \cap U_2$. Thus by exactness of the sequence there is a section $u \in \mathcal{F}_1(U_1 \cap U_2)$ such that $\alpha(u) = t_1 - t_2$ on $U_1 \cap U_2$. Since $\mathcal{F}_1$ is flabby, there exists a section $u' \in \mathcal{F}_1(U_2)$

with $u'|_{U_1 \cap U_2} = u$. Then $(t_2 + \alpha(u'))|_{U_1 \cap U_2} = t_1|_{U_1 \cap U_2}$ and hence the sections t_1 and $t_2 + \alpha(u')$ glue to give a section of $\mathcal{F}_2(U_1 \cup U_2)$.

Now let $s \in \mathcal{F}_3(U)$. Since β is surjective, therere exists an open cover $\{U_i\}$ of U, and sections $t_i \in \mathcal{F}_2(U_i)$ such that $\beta(t_i) = s|_{U_i}$. Pick a U_i, say U_0, and let Σ be the set of open subsets V of U which have sections t with $\beta(t) = s|_V$ and $t|_{U_0} = t_0$. Note $U_0 \in \Sigma$, so non-empty set. Also, every chain in Σ has an upper bound: if $V_0 \subseteq V_1 \subseteq \cdots$ with sections $t_i \in \mathcal{F}_2(V_i)$, then we can apply the result of the first paragraph repeatedly, i.e., t_1 can be modified so that t_0 and t_1 glue, t_2 can be modified so it glues to this section on $V_0 \cup V_1$, etc. Thus we can assume the sections t_i satisfy $t_i|_{V_j} = t_j$ for $j < i$, and hence by the sheaf gluing axiom, all these sections glue to give a section of $V = \bigcup_i V_i$, and hence V is an upper bound.

Thus by Zorn's lemma, Σ has a maximal element, call it V. Suppose $U \neq V$. Then there exists some $U_j \in \{U_i\}$ such that $U_j \not\subseteq V$. Then $\beta(t - t_j) = 0$ on $V \cap U_j$, so we can again apply the argument of the first paragraph to modify t_j so that t and t_j glue, contradicting maximality of V. Thus $U = V$, and t is the desired section of $\mathcal{F}_2$ over U with $\beta(t) = s$.

(b) Applying (a) to two open sets $V \subseteq U$, one obtains a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}_1(U) & \longrightarrow & \mathcal{F}_2(U) & \longrightarrow & \mathcal{F}_3(U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_1(V) & \longrightarrow & \mathcal{F}_2(V) & \longrightarrow & \mathcal{F}_3(V) & \longrightarrow & 0
\end{array}$$

As the first two vertical arrows are surjective since $\mathcal{F}_1$ and $\mathcal{F}_2$ are flabby, the third verical arrow is surjective by the snake lemma.

(c) Let $d_i : \mathcal{F}_i \to \mathcal{F}_{i+1}$ and let $\mathcal{Z}_i = \ker d_i = \operatorname{im} d_{i-1}$. This gives exact sequences

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{Z}_2 \to 0$$

and

$$0 \to \mathcal{Z}_i \to \mathcal{F}_i \to \mathcal{Z}_{i+1} \to 0$$

for $i \geq 2$. (note $\mathcal{Z}_1 = \mathcal{F}_0$ and $\mathcal{Z}_0 = 0$.) The first exact sequence tells us $\mathcal{Z}_2$ is flabby by (b), and the second exact sequence used repeatedly tells us all $\mathcal{Z}_i$ are flabbly. Thus we get short exact sequences after taking global sections, by (a), and these can then be reassembled into the desired long exact sequence.

(d) This is Hartshorne, Chapter III, Lemma 4.2.

(e) It is easy to see that a direct product of flabby sheaves is flabby, as a direct product of surjective maps is surjective. So it is sufficient to show that

$\mathcal{G} := \mathcal{F}|_{U_{i_0}\cap\cdots\cap U_{i_p}}$ is flabby. Here, we view this as a sheaf on X by push-forward, as usual. Then for $V \cap U \subseteq X$, $\mathcal{G}(U) = \mathcal{F}(U \cap U_{i_0} \cap \cdots \cap U_{i_p})$, $\mathcal{G}(V) = \mathcal{F}(V \cap U_{i_0} \cap \cdots \cap U_{i_p})$, so flabbiness of $\mathcal{G}$ follows from flabbiness of $\mathcal{F}$. Now taking global sections of the exact sequence of (d), we get by (c) the exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{F}).$$

Since this sequence is exact, we immediately conclude $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$.

4. Set $V_1 = D(x)$ and $V_2 = D(y)$, both open subsets of $X = \mathbb{A}^2_k$. Then $V_1 \cup V_2 = U$, so this forms a cover of U. Further, both V_1, V_2 and $V_1 \cap V_2$ are affine (being $\operatorname{Spec} k[x,y]_x$, $\operatorname{Spec} k[x,y]_y$ and $\operatorname{Spec} k[x,y]_{xy}$ respectively). So we may use this cover to compute Čech cohomology, and this will coincide with sheaf cohomology. We have the Čech 0-cohains

$$C^0 = k[x,y]_x \oplus k[x,y]_y$$

and 1-cochains

$$C^1 = k[x,y]_{xy}$$

with differential $d(f_1, f_2) = f_2 - f_1$, here using the natural inclusions of $k[x,y]_x$ and $k[x,y]_y$ into $k[x,y]_{xy}$. As the latter ring has a k-basis given by $\{x^i y^j \mid i, j \in \mathbb{Z}\}$, $k[x,y]_x$ has a k-basis given by $\{x^i y^j \mid i \in \mathbb{Z}, j \in \mathbb{Z}_{\geq 0}\}$, and $k[x,y]_y$ has a k-basis given by $\{x^i y^j \mid i \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}\}$, the cokernel of the differential clearly has a basis $\{x^i y^j \mid i, j \in \mathbb{Z}_{<0}\}$, as claimed.

5. The fact that U, V cover X is immediate, as $\mathbb{P}^2 \setminus (D_+(x_1) \cup D_+(x_2)) = \{(1,0,0)\} \not\subset X$. Note the fact that $(1,0,0) \notin X$ implies that f contains a monomial cx_0^d, which we can assume has coefficient $c = 1$.

Let $S = k[x_0, x_1, x_2]$ be the homogeneous coordinate ring of $\mathbb{P}^2$, $A = S/(f)$ the homogeneous coordinate ring of X, each viewed as graded rings.

Now

$$\Gamma(U, \mathcal{O}_X) = A_{(x_1)}, \quad \Gamma(V, \mathcal{O}_X) = A_{(x_2)},$$

where as usual the subscript (x_1) etc. means localization at x_1 in degree 0, while

$$\Gamma(U \cap V, \mathcal{O}_X) = A_{(x_1 x_2)},$$

with the restriction maps the obvious ones. It is convenient to consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S(-d)_{x_1} \oplus S(-d)_{x_2} & \xrightarrow{m_f} & S_{x_1} \oplus S_{x_2} & \longrightarrow & A_{x_1} \oplus A_{x_2} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_0^1} & & \downarrow{\scriptstyle d_0^2} & & \downarrow{\scriptstyle d_0^3} & & \\
0 & \longrightarrow & S(-d)_{x_1x_2} & \xrightarrow[m_f]{} & S_{x_1x_2} & \longrightarrow & A_{x_1x_2} & \longrightarrow & 0
\end{array}$$

Here the notation $S(-d)$ is a standard one which does book-keeping on degrees: $S(-d)$ is the graded S-module with $S(-d)_n = S_{-d+n}$. Thus multiplication by f is a degree 0 map with this convention, and we write multiplication by f in any of these cases as m_f. The vertical maps d_0^i are the Čech coboundary maps in each case, i.e., $(p,q) \mapsto p - q$, under the obvious inclusions $S_{x_1} \hookrightarrow S_{x_1x_2}$ etc. By the snake lemma, we then have a long exact sequence

$$0 \longrightarrow \ker d_0^1 \xrightarrow{m_f} \ker d_0^2 \longrightarrow \ker d_0^3 \longrightarrow \operatorname{coker} d_0^1 \xrightarrow{m_f} \operatorname{coker} d_0^2 \longrightarrow \operatorname{coker} d_0^3 \longrightarrow 0$$

where all maps are homogeneous of degree 0. We wish to compute the degree zero part of $\ker d_0^3$ and $\operatorname{coker} d_0^3$, being $H^0(X, \mathcal{O}_X)$ and $H^1(X, \mathcal{O}_X)$ respectively.

It is immediate that $\ker d_0^1$ is the intersection of $S(-d)_{x_1}$ and $S(-d)_{x_2}$ in $S(-d)_{x_1x_2}$, which is $S(-d)$. If we show that $m_f : \operatorname{coker} d_0^1 \to \operatorname{coker} d_0^2$ is injective in degree 0, then we see immediately that in degree 0, $\ker d_0^3 = \operatorname{coker}(\ker d_0^1 \to \ker d_0^2)$, so that $H^0(X, \mathcal{O}_X)$ is the degree 0 part of $S/(f) = A$, which is the field k.

Note that the image of d_0^2 is generated by monomials of the form $x_0^a x_1^b x_2^c$ with $a \in \mathbb{Z}_{\geq 0}$, and either $b \in \mathbb{Z}$, $c \in \mathbb{Z}_{\geq 0}$ or $b \in \mathbb{Z}_{\geq 0}$, $c \in \mathbb{Z}$. Thus the cokernel of d_0^2 has a basis of monomials of the form $x_0^a x_1^b x_2^c$ with $a \geq 0$, $b, c \leq -1$. The same is true for cokernel of d_0^1, with the shift in degree. In particular, the degree 0 part of $\operatorname{coker} d_0^1$ has a monomial basis $\{x_0^{b+c-d} x_1^{-b} x_2^{-c} \mid b, c \geq 1, b+c-d \geq 0\}$, while the degree 0 part of $\operatorname{coker} d_0^2$ has a monomial basis $\{x_0^{b+c} x_1^{-b} x_2^{-c} \mid b, c \geq 1, b+c \geq 0\}$. Now suppose given an element $\alpha \in \operatorname{coker} d_0^1$ of degree 0 with $m_f(\alpha) = 0$. Recall f has a term x_0^d. Let e be the largest power of x_0 occuring in any monomial in α. Let α' be a sum of those monomials in α for which this power is achieved, so we can write $\alpha' = x_0^e \alpha''$ with α'' not divisible by e. Then the sum of those monomials in $f\alpha$ with the largest power of x_0 is $x_0^{d+e}\alpha''$, and there is no possible cancellation with lower degree terms in x_0. Thus $m_f(\alpha) = 0$ implies $x_0^{d+e}\alpha'' = 0$ which implies $\alpha' = 0$. Thus $\alpha = 0$. This shows that $H^0(X, \mathcal{O}_X) \cong k$.

Now consider the degree 0 part of the cokernel of $m_f : \operatorname{coker} d_0^1 \to \operatorname{coker} d_0^2$: this will give $H^1(X, \mathcal{O}_X)$. First note that the image in $\operatorname{coker} m_f$ of any monomial of the form $x_0^{a+b}x_1^{-a}x_2^{-b}$ representing an element of $\operatorname{coker} d_0^2$ with $a+b \geq d$ can be written in terms of monomials with smaller x_0 power: indeed, $f \cdot x_0^{a+b-d}x_1^{-a}x_2^{-b}$ is in the image of m_f, and this is of the form $x_0^{a+b}x_1^{-a}x_2^{-b} + \cdots$ where the dots represents terms with lower degree in x_0. Thus every element of the cokernel is represented as a sum of monomials with x_0 degree $a+b < d$. Further, any element of the image of m_f must have a monomial of the form $x_0^{a+b}x_1^{-a}x_2^{-b}$ with $a+b \geq d$, as f has the monomial x_0^d. Thus in $\operatorname{coker} m_f$, there are no k-linear relations between the monomials

$$\{x_0^{a+b}x_1^{-a}x_2^{-b} \,|\, a+b < d, a \geq 1, b \geq 1\}.$$

Thus this set forms a basis for the degree zero part of the cokernel of m_f. Thus the dimension is $\sum_{a=1}^{d-2} d-1-a = \sum_{a=1}^{d-2} a = (d-2)(d-1)/2$, as desired.

Remark. The exercise asked you to carry out a direct Čech calculation. Ultimately, what the above is doing is really asking you to get your hands a bit dirty with the Čech calculation of the cohomology of $\mathbb{P}^2$. A much quicker way to carry out this calculation, once one knows the cohomology of line bundles on $\mathbb{P}^n$, is to observe we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_X \to 0,$$

as $\mathcal{O}_{\mathbb{P}^2}(-d)$ is isomorphic to the ideal sheaf of X in $\mathbb{P}^2$. Taking the long exact sequence of cohomology, using the calculation of cohomology of line bundles on $\mathbb{P}^2$, gives sequences

$$0 \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to H^0(X, \mathcal{O}_X) \to 0$$

and

$$0 \to H^1(X, \mathcal{O}_X) \to H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d)) \to 0,$$

from which the exercise follows much more efficiently, as $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d)) \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3))$.

6. (a) We have a homomorphism $B \to B \otimes_A B$ given by $b \mapsto b \otimes 1$, giving $B \otimes_A B$ a B-algebra structure, and hence I/I^2 has a B-module structure.

First we check the given d is a derivation. (1) If $b, b' \in B$, then $d(b+b') = 1 \otimes (b+b') - (b+b') \otimes 1 = d(b) + d(b')$. (2) $d(bb') = 1 \otimes (bb') - (bb') \otimes 1 = (b \otimes 1)(1 \otimes b' - b' \otimes 1) + (1 \otimes b - b \otimes 1)(1 \otimes b')$. Now working modulo I^2, we note that $(1 \otimes b' - b' \otimes 1)(1 \otimes b - b \otimes 1) \equiv 0 \mod I^2$, so we can subtract this from the above expression, and see that $d(bb') = bd(b') + b'd(b)$, as desired. (3) $d(a) = 1 \otimes a - a \otimes 1 = 1 \otimes a - 1 \otimes a = 0$, as the the tensor product is over A.

Now suppose given a B-module M and an A-derivation $D : B \to M$. There is an A-bilinear map $B \times B \to M$ given by $(b, b') \mapsto b \cdot D(b')$. Thus by the universal property of tensor product, this induces a map $f : B \otimes_A B \to M$, $b \otimes b' \mapsto b \cdot D(b')$. We may restrict f to I, to get a homomorphism of B-modules $f : I \to M$. Note that $f(1 \otimes b - b \otimes 1) = D(b)$.

I now claim that I is generated as a B-module by elements of the form $d(b)$. Indeed, writing $I \ni \sum b_i \otimes b'_i = \sum b_i(1 \otimes b'_i - b'_i \otimes 1)$ as $\sum b_i b'_i = 0$, we immediately see this.

Thus I^2 is generated by elements of the form $(1 \otimes b - b \otimes 1)(1 \otimes b' - b' \otimes 1)$, and applying f to this gives, after expanding, $D(bb') - bD(b') - b'D(b) + bb'D(1)$. Since $D(1) = 0$ and D is a derivation, this is 0. Thus f is zero on I^2 and hence induces a map $f : I/I^2 \to M$. Clearly $D = f \circ d$ by construction. Further, this choice of f is unique because f is determined on the elements $d(b)$, and these elements generate I/I^2.

(b) While one can compute directly from the above construction, it is easier to just show the asserted description of $\Omega_{A/k}$ satisfies the universal property. The given d is clearly a derivation, via the product rule for differentiation. Given $D : A \to M$ a k-derivation for an A-module M, define $f : \Omega_{A/k} \to M$ as an A-module homomorphism by $f(dx_i) = D(x_i)$. Then for a polynomial $p \in A$, repeated use of the Leibniz rule shows that $D(p) = \sum_i (\partial p/\partial x_i) D(x_i)$, from which it immediately follows that $D = f \circ d$ and f is the unique such map satisfying this.

(c) Let $X = \mathbb{A}^n_k$. Recall that the sheaf of differentials was defined to be the conormal bundle $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal of the diagonal embedding of X in $X \times_k X$. With A as in part (b), note that $X \times_k X = \operatorname{Spec} A \otimes_k A$, the diagonal morphism $\Delta : X \to X \times_k X$ is induced by $A \otimes_k A \to A$ taking $a \otimes 1 \mapsto a$, $1 \otimes a \mapsto a$, which is the map introduced in part (a). The ideal in $A \otimes_k A$ defining the closed subscheme $\Delta(X) \subseteq X \times_k X$ is precisely I, i.e., $\Gamma(X \times_k X, \mathcal{I}) = I$. Similarly, $\Gamma(X \times_k X, \mathcal{I}^2) = I^2$. Using problem 1 above, one concludes that $\Gamma(X \times_k X, \mathcal{I}/\mathcal{I}^2) = I/I^2 = \Omega_{A/k}$. Thus $dx_1, \ldots, dx_n$ form a basis of global sections of $\Omega_{X/k}$. We stated in lecture that $\Omega_{X/k}$ is locally free of rank $n = \dim X$, and the dx_i are easily checked to be linearly independent on stalks (the stalk of $\Omega_{X/k}$ agrees with $\Omega_{A_\mathfrak{p}/k}$ at a point $\mathfrak{p} \in \operatorname{Spec} A$, as can easily be checked), and thus the dx_i generate $\Omega_{X/\Bbbk}$ as a free sheaf of $\mathcal{O}_X$-modules.

7. First note in general that if X is genus g, then Riemman-Roch says

$$\dim H^0(X, \omega_X) - \dim H^0(X, \omega_X \otimes \omega_X^{-1}) = \deg K_X + 1 - g.$$

As $\dim H^0(X, \omega_X) = g$ by definition, $\dim H^0(X, \mathcal{O}_X) = 1$, (the only global regular functions on X are constant), one gets $\deg K_X = 2g - 2$. Thus in the case $g = 2$, $\deg K_X = 2$. Applying Riemann-Roch to D and noting that $\dim H^0(X, \mathcal{O}_X(K_X - D)) = 0$ as $\deg K_X - D < 0$, we see that

$$\dim H^0(X, \mathcal{O}_X(D)) = \deg D + 1 - g = 4.$$

Thus the linear system $|D|$ is dimension three, and hence, using the fact D is very ample, we obtain an embedding $f : X \to \mathbb{P}^3$.

We now have an exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$$

and after tensoring with $\mathcal{O}_{\mathbb{P}^3}(n)$, we get the exact sequence

$$0 \to \mathcal{I}_X(n) \to \mathcal{O}_{\mathbb{P}^3}(n) \to \mathcal{O}_X(n) \to 0$$

(where the sheaves on the left and right are defined using this tensor product). Exactness on the left follows from $\mathcal{O}_{\mathbb{P}^3}(n)$ being locally free.

Note that $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n))$ is the space of homogeneous polynomials in four variables of degree n, and this space has dimension $\binom{n+3}{3}$. Note further that $H^0(\mathbb{P}^3, \mathcal{I}_X(n)) = \ker H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to H^0(\mathbb{P}^3, \mathcal{O}_X(n))$, and the latter morphism is given by restriction. Thus this kernel can be viewed as the vector space of homogeneous polynomials of degree n which vanish on X. Further, we know the dimension of $H^0(\mathbb{P}^3, \mathcal{O}_X(n))$ for $n \geq 1$ by Riemann-Roch. Note here use the convention that a sheaf on X is viewed as a sheaf on $\mathbb{P}^3$ via push-forward, but we don't write the push-forward explicitly. So in particular, $H^0(\mathbb{P}^3, \mathcal{O}_X(n)) = H^0(X, \mathcal{O}_X(nD))$, and as $\deg K_X - nD < 0$ for $n \geq 1$, we have

$$\dim H^0(X, \mathcal{O}_X(nD)) = 5n - 1$$

by Riemann-Roch.

Now necessarily X is not contained in a plane in $\mathbb{P}^3$, as otherwise $|D|$ would be a two-dimensional linear system (alternatively, X would be genus 6 by question 5). Thus $H^0(\mathbb{P}^3, \mathcal{I}_X(1)) = 0$. On the other hand, consider the dimensions of the vector spaces in the exact sequence

$$0 \to H^0(\mathbb{P}^3, \mathcal{I}_X(2)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathbb{P}^3, \mathcal{O}_X(2)).$$

By the above discussion, the second and third spaces have dimensions $\binom{5}{2} = 10$ and $5 \cdot 2 - 1 = 9$ respectively. Thus the kernel is at least one-dimensional, so there is at least one quadric surface containing X. On the other hand, there can't be two distinct quadric surfaces containing X, as, since X is not contained in a plane, each quadric surface must be irreducible, and the intersection of two irreducible quadric surfaces is a curve of degree 4 by Bézout's theorem. Thus X is contained in a unique irreducible quadric surface.

Now consider similarly

$$0 \to H^0(\mathbb{P}^3, \mathcal{I}_X(3)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathbb{P}^3, \mathcal{O}_X(3)).$$

The second and third spaces have dimensions 20 and 14 respectively, so

$$\dim H^0(\mathbb{P}^3, \mathcal{I}_X(3)) \geq 6.$$

Furthermore, if f_2 is the equation of the quadric surface containing X, then certainly $f_2 x_0, \ldots, f_2 x_3$ are linearly independent cubic polynomials vanishing on X, and any reducible cubic surface containing X necessarily has equation lying in the span of these four polynomials. Since $6 > 4$, there must be an irreducible cubic containing X, as desired.