Solutions to Example Sheet 3.

1. Take any open affine subset $U \subseteq X$ with $Z \cap U \neq \emptyset$. If $U = \operatorname{Spec} A$, then $Z \cap U = V(I)$ for some ideal $I \subseteq A$. Since $V(I) = V(\sqrt{I})$, we can assume I is radical. If I is not prime, then one can find $f, g \in A \setminus I$ with $fg \in I$, and then $(V(f) \cap V(I)) \cup (V(g) \cap V(I)) = (V(f) \cup V(g)) \cap V(I) = V(fg) \cap V(I) = V(I)$, but neither $V(f) \cap V(I), v(g) \cap V(I)$ are $V(I)$, violating irreducibility of Z. Thus I is prime, corresponding to a point $\eta \in U$. Then $\overline{\{\eta\}} = V(I)$ in U, and since Z is irreducible, $Z \cap U$ is dense in Z, so $\overline{\{\eta\}} = Z$ in X.

To show η is unique, if there exists $\eta, \eta' \in Z$ with $\overline{\{\eta\}} = \overline{\{\eta'\}}$, then necessarily $\eta' \in U$ also, and thus $\eta' = \eta$ since $\overline{\{\eta'\}} = V(I)$ in U.

2. (a) Let $x \in X$ be a closed point. As X is a variety, x has an open affine neighbourhood $U = \operatorname{Spec} A$ with A a finitely generated k-algebra and with x corresponding to a maximal ideal $\mathfrak{m}$. Then $\dim A_{\mathfrak{m}} = \operatorname{height}(\mathfrak{m})$ (via one-to-one correspondence between prime ideals contained in $\mathfrak{m}$ and prime ideals of $A_{\mathfrak{m}}$), and $\operatorname{height}(\mathfrak{m}) + \dim A/\mathfrak{m} = \dim A$. As $A/\mathfrak{m}$ is a field, it has dimension zero, so we can conclude that $\dim A = \dim A_{\mathfrak{m}}$. Thus $\dim U = \dim A = \dim A_{\mathfrak{m}}$, by the one-to-one correspondence between irreducible closed subsets of U and prime ideals of A.

Now suppose given any two affine open subsets $U = \operatorname{Spec} A$, $U' = \operatorname{Spec} A'$ of X, with both A, A' finitely generated k-algebras. Then $\dim U = \dim A = \text{tr.deg.}(A_{(0)}/k) = \text{tr.deg.}(A'_{(0)}/k) = \dim U'$, as $A_{(0)} = A'_{(0)} = K(X)$. Thus U, U' have the same dimension.

In general, if $\{U_i\}$ is an affine open cover of any scheme X, then $\dim X = \sup_i \dim U_i$. Indeed, if $Z_0 \subsetneq \cdots \subsetneq Z_n$ is a chain of irreducible closed subsets of X, then $Z_0 \cap U_i \neq \emptyset$ for some i, so $Z_0 \cap U_i \subsetneq \cdots \subsetneq Z_n \cap U_i$ is a chain of irreducible closed subsets of U_i, with the inequalities holding as the closure of $Z_j \cap U_i$ is Z_j. Thus $\dim X \leq \sup_i \dim U_i$. Conversely, given a chain of irreducible closed subsets in U_i, we get a chain of irreducible closed subsets in X by taking closures, so $\dim X \geq \dim U_i$. This gives the claim.

In our particular situation, we can choose a cover of X by spectra U_i of finitely generated k-algebras, all of the same dimension by the first two paragraphs. Thus $\dim X = \dim U_i = \dim \mathcal{O}_{X,x}$ for any closed point x.

(b) Write $Y = \bigcup_i Y_i$ for the decomposition of Y into irreducible closed sets, and let η_i be the generic point of Y_i, guaranteed by Question 2. Then $\operatorname{codim}(Y, X) = \inf_{Z \subseteq Y} \operatorname{codim}(Z, X)$ by definition, where the infinum is over all irreducible closed subsets of Y. Since any irreducible closed subset is contained in one of the Y_i's, this agrees with $\inf_i \operatorname{codim}(Y_i, X)$, and thus it is enough to show that $\operatorname{codim}(Y_i, X) = \dim \mathcal{O}_{X,\eta_i}$. (Here we use the fact that if $x \in \overline{\{\eta_i\}}$, then $\mathcal{O}_{X,\eta_i}$ is a localization of $\mathcal{O}_{X,x}$, so $\dim \mathcal{O}_{X,\eta_i} \leq \dim \mathcal{O}_{X,x}$.) Now let U be an open affine set containing η_i. Then a chain $Y_i = Z_0 \subsetneq \cdots \subsetneq Z_n \subseteq X$ of irreducible closed subsets induces a chain of irreducible closed subsets $U \cap Y_i = U \cap Z_0 \subsetneq \cdots \subsetneq U \cap Z_n$ and vice versa, so $\operatorname{codim}(Y_i, X) = \operatorname{codim}(Y_i \cap U, U)$. On the other hand, if $U = \operatorname{Spec} A$, η_i corresponds to a prime $\mathfrak{p}$, then clearly $\operatorname{codim}(Y_i \cap U, U) = \operatorname{ht} \mathfrak{p}$. But $\operatorname{ht} \mathfrak{p} = \dim A_{\mathfrak{p}} = \dim \mathcal{O}_{X,\eta_i}$, hence the result. [Note: We have not used an properties of finitely generated k-algebras which are dmains here, so this result holds for all schemes.]

(c) First assume Y is irreducible, with generic point η. Let $U \subseteq X$ be an open affine subset, $U = \operatorname{Spec} A$ with A a finitely generated k-algebra, with $U \cap Y \neq \emptyset$. Then $U \cap Y = V(\mathfrak{p})$ for some prime $\mathfrak{p} \subseteq A$, and $\dim Y = \dim U \cap Y = \dim A/\mathfrak{p} = \dim A - \operatorname{ht} \mathfrak{p} = \dim X - \operatorname{codim}(Y, X)$ by the discussion of (a) and (b). Now if $Y = \bigcup_i Y_i$ is a decomposition into irreducible components, we have $\dim Y = \sup \dim Y_i$ and $\operatorname{codim}(Y, X) = \inf \operatorname{codim}(Y_i, X)$, and since $\dim Y_i + \operatorname{codim}(Y_i, X) = \dim X$, $\dim Y_i$ achieves the supremum if $\operatorname{codim}(Y_i, X)$ achieves the infinum, and the result follows.

(d) Cover U with open affines, and cover each of these with open affines which are finitely generated over k (which we can do by the finite type assumption and Question 2 on Example Sheet II). Let $\{U_i\}$ be this open affine cover. By the argument in (a), $\dim U = \sup_i \dim U_i = \dim U_i = \dim X$.

(e) If $U \subseteq X$ is open affine with $U = \operatorname{Spec} A$ with A a finitely generated k-algebra, then $\dim X = \dim U = \dim A = \text{tr.deg.} A_{(0)}/k = \text{tr.deg.} K(X)/k$. The first equality is from (d), and the last is since $A_{(0)} = K(X)$.

3. (a) Let $X = \operatorname{Spec} k[x, y, z]/(xz, yz)$. Note that X has two irreducible components, $V(x, y)$ and $V(z)$, of dimensions one and two respectively. Thus $\dim X = 2$, but if $\mathfrak{p} = (x, y, z - 1)$ then $\mathcal{O}_{X,\mathfrak{p}} = k[z]_{(z-1)}$ is one-dimensional, and $\mathfrak{p}$ is a maximal ideal.

(c) Continuing with the same example, if Y is taken to be the point $\{\mathfrak{p}\}$ as above, then $\dim Y = 0$, $\operatorname{codim}(Y, X) = 1$, $\dim X = 2$, contradicting (c) in Question 3.

(d) Let $R = k[x]_{(x)}$. Then R has two prime ideals, (0) and (x), with the former the generic point and (x) the closed point. Then $\dim X = 1$ with $X = \operatorname{Spec} R$, and if $U = X \setminus \{(x)\}$, then U is an open subset of dimension 0.

4. (a) With $U := \mathbb{A}^1_k \times \mathbb{A}^1_k = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$, we have $\operatorname{Cl}(U) = 0$. Now we have inclusions $U \subset \mathbb{P}^1_k \times \mathbb{A}^1_k \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$, using your favorite standard open affine $\mathbb{A}^1 \subseteq \mathbb{P}^1$. [We follow the usual convention that we write $\times$ rather than $\times_{\operatorname{Spec} k}$ when working with schemes defined over $\operatorname{Spec} k$.] By twice making use of the exact sequence $\mathbb{Z} \to \operatorname{Cl} X \to \operatorname{Cl} U \to 0$ proved in class, we see we have a surjection $\mathbb{Z}^2 \to \operatorname{Cl} \mathbb{P}^1 \times \mathbb{P}^1$, with $(1,0) \mapsto \ell = \mathbb{P}^1 \times \{\infty\}$ and $(0,1) \mapsto m = \{\infty\} \times \mathbb{P}^1$, where ∞ is the unique point of $\mathbb{P}^1 \setminus \mathbb{A}^1$. Thus $\operatorname{Cl} X$ is generated by ℓ and m and we just need to show there are no relations. But suppose there is a rational function $f \in K(X) = K(U)$ with $(f) = a\ell + bm$. Then writing $f = f_1(x,y)/f_2(x,y)$ for $f_1, f_2 \in k[x,y]$ relatively prime, f necessarily vanishes along the curve $f_1 = 0$ and has poles along $f_2 = 0$. In order for $(f) = 0$ on $\mathbb{A}^2$, we would thus have to have $f \in k$. But then $(f) = 0$ on X also. Thus there are no relations and $\operatorname{Cl} X = \mathbb{Z}\ell \oplus \mathbb{Z}m$.

(b) Let Y be the prime divisor $V(x,z) \subseteq X$. Note that $V(x) = V(x,z)$, but these are different ideals. However, this does tell us that $U := X \setminus Y = D(x) = \operatorname{Spec}(k[x,y,z]/(xy - z^2))_x = \operatorname{Spec} k[x,z]_x$. (To see this, note that once we localize at x, we can eliminate the variable y as $y = z^2/x$ in the localized ring.) Now $k[x,z]_x$ is a UFD, so $\operatorname{Cl} U = 0$, and $\operatorname{Cl} X$ is thus generated by the prime divisor Y. We only need to determine the relations.

Suppose there is a rational function f with $(f) = aY$ for some $a \neq 0$. Since on U, we would have $(f) = 0$, f is then a regular invertible function on U. However, the only invertible functions on U are the units in $k[x,z]_x$, which are of the form cx^n for $c \in k^*$. Thus we calculate the divisor of zeros and poles (x) of x on X, it being of the form bY for some b. Now the stalk of $\mathcal{O}_X$ at the generic point of Y is the localized ring $(k[x,y,z]/(xy - z^2))_{(x,z)}$, and since $y \notin (x,z)$, y is invertible and we can eliminate $x = z^2/y$, so this ring is isomorphic to $k[y,z]_{(z)}$. Now clearly $\nu_Y(x) = \nu_Y(z^2/y) = 2$. Thus $(x) = 2Y$, $(cx^n) = 2nY$, and we see $\operatorname{Cl} X = \mathbb{Z}/2\mathbb{Z}$.

(c) Let $Y_1 = V(x,z) \subseteq X$; again this is a prime divisor, as (x,z) is a prime ideal in $k[x,y,z,w]/(xy - zw)$. (You just check this by quotienting out by this ideal and you get $k[y,w]$.) Similarly, let $Y_2 = V(x,w)$. Then $Y_1 \cup Y_2 = V(x)$ and $U = X \setminus (Y_1 \cup Y_2) = D(x) \cong \operatorname{Spec}(k[x,y,z,w]/(xy - zw))_x = \operatorname{Spec} k[x,z,w]_x$ pretty much as in (b). Thus $\operatorname{Cl} U = 0$ as $k[x,z,w]_x$ is a UFD, and so $\operatorname{Cl} X$ is generated by Y_1 and Y_2. On the other hand, the divisor of zeros and poles of x is $(x) = Y_1 + Y_2$; this can be checked exactly as in (b). Thus there is a relation $Y_1 \sim -Y_2$. We need to check there is no further relation. Suppose $aY_1 + bY_2 \sim 0$. Using $Y_1 \sim -Y_2$, we would also get a relation $a'Y_1 + b'Y_2 \sim 0$ with $a', b' \geq 0$. Thus there must be a *regular* function f with $(f) = a'Y_1 + b'Y_2$. Such a function is then invertible on U, and the group of units of $k[x,z,w]_x$ consists of monomials of the form cx^n for $c \in k^*$ nad $n \in \mathbb{Z}$. However, $(cx^n) = (x^n) = n(x) = nY_1 + nY_2$. Hence any relation is a multiple of $Y_1 + Y_2 = 0$. Thus $\mathrm{Cl}\, X = \mathbb{Z}$, generated by $Y_1 \sim -Y_2$.

5. A morphism $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ is induced by a surjection $\mathcal{O}_{\mathbb{P}^n}^{\oplus(m+1)} \to \mathcal{L}$ for some line bundle $\mathcal{L}$ on $\mathbb{P}^n$. Necessarily, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(r)$ for some r, again necessarily $r \geq 0$ as $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = 0$ for $r < 0$. Such a morphism is then given by a choice of $m+1$ sections $s_0, \ldots, s_m$ of $\mathcal{L}$ which generate $\mathcal{L}$ globally, i.e., at each point $x \in \mathbb{P}^n$, at least one of the s_i's is non-vanishing, i.e., $s_i \notin \mathfrak{m}_x \mathcal{L}_x$. Note that the sections s_i correspond to homogeneous polynomials f_i of degree r, and the common vanishing locus of all f_i is $V(f_0, \ldots, f_m)$. However, if $m < n$ then these polynomials will have a common zero (either by dimension theory, or think of Bézout's theorem), and hence $s_0, \ldots, s_m$ cannot generate $\mathcal{L}$ globally. Thus $m \geq n$.

Now suppose $\dim \varphi(\mathbb{P}^n) < n \leq m$. Then one can find a point $x \in \mathbb{P}^m \setminus \varphi(\mathbb{P}^n)$, which induces a *linear projection* $\pi : \mathbb{P}^m \setminus \{x\} \to \mathbb{P}^{m-1}$. [Thinking of $\mathbb{P}^m$ as the set of one-dimensional vector spaces of a vector space V of dimension $m+1$, and thinking of x as corresponding to $W_x \subseteq V$ one-dimensional, π is induced by the projection $V \to V/W_x$. If you are a stickler for details, convince yourself this defines a morphism of schemes.] Now $\pi \circ \varphi : \mathbb{P}^n \to \mathbb{P}^{m-1}$ is a morphism. We can thus continue decreasing the dimension of the target space until $m < n$, a contradiction.

6. If M is an A-module, one sees immediately from the construction of the sheaf $\widetilde{M}$ that $\widetilde{M}|_{D(f)} = \widetilde{M_f}$. Thus if $\mathcal{F}$, $\mathcal{G}$ are two quasi-coherent sheaves on X, we can find a single open affine cover $\{U_i = \mathrm{Spec}\, A_i\}$ such that $\mathcal{F}|_{U_i} = \widetilde{M}_i$, $\mathcal{G}|_{U_i} = \widetilde{N}_i$ for A_i-modules M_i, N_i. This makes use of Example Sheet II, Question 2.

Note taking twiddles is a functor, so a module homomorphism $M_i \to N_i$ induces a morphism of sheaves of $\mathcal{O}_{\mathrm{Spec}\, A_i}$-modules $\widetilde{M}_i \to \widetilde{N}_i$. Conversely, such a morphism induces an A_i-module homomorphism by taking global sections, and it is easy to check that this identifies $\mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec}\, A_i}}(\widetilde{M}_i, \widetilde{N}_i)$ and $\mathrm{Hom}_{A_i}(M_i, N_i)$.

Thus in particular, we may reduce to the case that $X = \mathrm{Spec}\, A$ and $\mathcal{F} = \widetilde{M}$, $\mathcal{G} = \widetilde{N}$, and $\tilde{f} : \mathcal{F} \to \mathcal{G}$ is induced by $f : M \to N$. Here M, N are A-modules.

Note next that taking twiddles is an exact functor, as $(\widetilde{M})_{\mathfrak{p}} = M_{\mathfrak{p}}$ and localization is an exact functor, while a sequence of maps of sheaves is exact if and only if it is exact on stalks. From this immediately follows that

$$\ker \tilde{f} = \widetilde{\ker f}$$
$$\operatorname{coker} \tilde{f} = \widetilde{\operatorname{coker} f}$$
$$\operatorname{im} \tilde{f} = \widetilde{\operatorname{im} f}$$

This shows that kernels, cokernels and images of morphisms of quasi-coherent sheaves are quasi-coherent. The statement in the coherent case follows from the fact that if A is Noetherian, then a submodule of a finitely generated module is finitely generated (and of course in general a quotient module of a finitely generated module is finitely generated).

Given $f : X \to Y$, we can cover Y with open affines U on which $\mathcal{F}$ is given as $\widetilde{M}$, and by replacing Y by U and X by $f^{-1}(U)$, we can assume $Y = \operatorname{Spec} B$ is affine and $\mathcal{F} = \widetilde{M}$ for M a B-module. Now there is a presentation

$$F_1 \to F_0 \to M \to 0$$

where $F_j = \bigoplus_{i \in I_j} A$ are free modules. Then taking twiddles gives an exact sequence

$$\bigoplus_{i \in I_1} \mathcal{O}_Y \to \bigoplus_{i \in I_0} \mathcal{O}_Y \to \mathcal{F} \to 0.$$

We may then apply f^*, noting that f^{-1} is an exact functor (as it preserves stalks as it clear from the definitions) while tensoring with $\mathcal{O}_X$ is a right exact functor, so that we get

$$\bigoplus_{i \in I_1} \mathcal{O}_X \to \bigoplus_{i \in I_0} \mathcal{O}_Y \to f^*\mathcal{F} \to 0.$$

Thus $f^*\mathcal{F}$ is described as the cokernel of a morphism between quasi-coherent sheaves, and hence is quasi-coherent. Note in particular this shows that

$$f^*\widetilde{M} = \widetilde{M \otimes_B A}.$$

For the counter example, consider $f : X = \mathbb{A}^1_k \to \operatorname{Spec} k$, $\mathcal{F} = \mathcal{O}_X$. Then $f_*\mathcal{O}_X$ is the k-vector space $k[x]$, if $\mathbb{A}^1_k = \operatorname{Spec} k[x]$. But $k[x]$ is not a finite dimensional k-vector space.

7. (a) As $i_U^\# : \mathcal{O}_X(U) \to (i_*\mathcal{O}_Z)(U) = \mathcal{O}_Z(Z \cap U)$ is a ring homomorphism, its kernel $\mathcal{I}(U)$ is an ideal in $\mathcal{O}_X(U)$, proving the claim.

(b) This statement follows from Question 6 if we know that $i_*\mathcal{O}_Z$ is coherent. In fact, it is not in particular difficult to show that if $f : X \to Y$ is a morphism and $\mathcal{F}$ is quasi-coherent on X, then $f_*\mathcal{F}$ is quasi-coherent on Y, which is actually sufficient for our purposes. In particular $i_*\mathcal{O}_Z$ would be quasi-coherent, and so the kernel of $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is quasi-coherent, and coherent if X is Noetherian, since it is a sub-$\mathcal{O}_X$-module of $\mathcal{O}_X$.

For pedagogical reasons, I will give a harder proof, reducing the statement to a statement that a closed subscheme of $\operatorname{Spec} A$ is always of the form $\operatorname{Spec} A/I$ for $I \subseteq A$ an ideal, and then proving this statement.

Let $U \subseteq X$ be an open affine subset, $U = \operatorname{Spec} A$. We first show that $Z \cap U \cong \operatorname{Spec} A/I$ for some ideal $I \subseteq A$. To this end, we can replace X by U and Z by $Z \cap U$, and assume that $X = \operatorname{Spec} A$ is affine. We have an induced map $\varphi := i_X^\# : A \to \Gamma(Z, \mathcal{O}_Z)$. Let $I := \ker \varphi$, an ideal in A. We wish to show $Z \cong \operatorname{Spec} A/I$ with the immersion $Z \to X$ induced by the quotient map $A \to A/I$.

Certainly φ factors through the quotient map $A \to A/I$, giving a factorization $Z \to \operatorname{Spec} A/I \to \operatorname{Spec} A$. Let us now replace $X = \operatorname{Spec} A$ with $\operatorname{Spec} A/I$, which allows us to assume that $\ker \varphi$ is zero, i.e., φ is injective. We now wish to show that in fact $i : Z \to X$ is an isomorphism.

We first show the underlying map i is a homeomorphism. We know that it is injective (being a closed immersion) and closed (i.e., closed sets are mapped to closed sets, again being a closed immersion). So we just need to show it is surjective. If it is not surjective, then, as $i(Z)$ is closed, there exists an $a \in A$ such that $Z \subset V(a) \neq \operatorname{Spec} A$. Now let $V \subseteq Z$ be an open affine subset $\operatorname{Spec} B$. We write $V'(J) \subseteq \operatorname{Spec} B$ for an ideal $J \subset B$ (to distinguish from other occurences of $V(\cdot)$ above). Then $V \subset i^{-1}(V(a)) \cap V = V'((\varphi(a)|_V))$. Thus $\varphi(a)|_V$ is nilpotent in B, so $\varphi(a^N)|_V = 0$ for some $N > 0$. By quasi-compactness, we can cover Z by a finite number of open affines of this form, so we can take N sufficiently large to work for all affines V in this cover. Thus $\varphi(a^N) = 0$ by the first sheaf axiom, and so $a^N = 0$ by injectivity of φ. From this we conclude that $a \in \sqrt{0}$ and $V((a)) = \operatorname{Spec} A$. Thus in fact Z is not contained in a proper closed subset of $\operatorname{Spec} A$, showing the desired surjectivity. Thus $Z \to X$ is a homeomorphism.

It remains to show that the homomorphism $i^\# : \mathcal{O}_X \to \mathcal{O}_Z$ is bijective. It is surjective by assumption, and we test injectivity on stalks. For $x \in X$, $\mathcal{O}_{X,x} = A_{\mathfrak{p}_x}$, where $\mathfrak{p}_x$ is the prime ideal corresponding to x. It is enough to show that every element of $\ker(\mathcal{O}_{X,x} \to \mathcal{O}_{Z,x})$ of the form $g/1 \in A_{\mathfrak{p}_x}$ is zero in this localization. Indeed, the kernel is an ideal, and if g/s lies in the kernel for $s \notin \mathfrak{p}_x$, then so does $g/1$, and s is a unit in the ring. Thus if $g/1 = 0$, $g/s = 0$ also. Given $g \in A$, cover $Z = U \cup \bigcup_{i \in I} U_i$ be an affine open cover of Z by a finite number of open sets (quasi-compactness again) such that $x \in U$, $x \notin U_i$ for any U_i, and $\varphi(g)|_U = 0$. (Such a U exists because $\varphi(g)/1$ is assumed to be zero in the stalk $\mathcal{O}_{Z,x}$).

Choose $s \in A$ with $x \in D(s) \subseteq U$. Note $s \notin \mathfrak{p}_x$. If we can show that $\varphi(s^N g) = 0$ for some N, then by injectivity of φ, $s^N g = 0$, and thus $g/1 = 0$ in $A_{\mathfrak{p}_x}$, as desired. But $\varphi(g)|_U = 0$ by assumption, so $\varphi(s^N g)|_U = 0$. Now $D_{U_i}(\varphi(s)|_{U_i}) = D(s) \cap U_i \subseteq U \cap U_i$ (here D_{U_i} denotes D for the affine open subset U_i), so we get $\varphi(g)|_{D_{U_i}(\varphi(s)|_{U_i})} = 0$. Thus, the image of $\varphi(g)$ in the localization $\Gamma(U_i, \mathcal{O}_Z)_{\varphi(s)|_{U_i}}$ is zero, i.e., $\varphi(s^N g)|_{U_i} = 0$ for some N. Taking N sufficiently large to work for all i, we get $\varphi(s^N g) = 0$ for some N. By injectivity of φ, we see $s^N g = 0$ so $g/1 = 0$ in $\mathcal{O}_{X,x}$, as desired.

Having now shown that i induces an isomorphism between Z and $\operatorname{Spec} A/I$, it is immediate that $i_*\mathcal{O}_Z = \widetilde{A/I}$, giving the desired coherence of $i_*\mathcal{O}_Z$.

c) We have already seen that a closed immersion $i : Z \hookrightarrow X$ gives rise to a quasi-coherent sheaf of ideals $\mathcal{I}$ with $i_*\mathcal{O}_Z \cong \mathcal{O}_X/\mathcal{I}$ by construction of $\mathcal{I}$. Conversely, suppose given $\mathcal{I}$ a quasi-coherent sheaf of ideals. Then consider

$$Z := \operatorname{supp} \mathcal{O}_X/\mathcal{I} := \{x \in X \mid (\mathcal{O}_X/\mathcal{I})_x \neq 0\}.$$

This is in fact a closed set. Indeed, $1 \in \Gamma(X, \mathcal{O}_X)$ must generate the stalk $(\mathcal{O}_X/\mathcal{I})_x$, and so the stalk is zero if and only if $1 = 0$ in this stalk. However, if this holds in the stalk, it also holds in an open neighbourhood of x, so the complement of $\operatorname{supp} \mathcal{O}_X/\mathcal{I}$ is open.

Now let $i : Z \hookrightarrow X$ be the inclusion, and set $\mathcal{O}_Z = i^{-1}(\mathcal{O}_X/\mathcal{I})$. We wish to show $(Z, \mathcal{O}_Z)$ is a closed subscheme of X. To do so, we may test this on open affines on which $\mathcal{I}$ is the twiddle of a module. So assume $X = \operatorname{Spec} A$, $\mathcal{I} = \widetilde{I}$. Note that as $\mathcal{I} \subseteq \mathcal{O}_X$, $I = \Gamma(X, \mathcal{I})$ is a sub-module of $\Gamma(X, \mathcal{O}_X) = A$, i.e., I is an ideal in A.

Let us write $\mathfrak{p}_x \in A$ for the prime ideal corresponding to $x \in A$. The localizations $I_{\mathfrak{p}_x}$ agree with the stalks $\mathcal{I}_x$ for $x \in X$. In particular, $x \in Z$ if and only if $I_{\mathfrak{p}_x} \neq A_{\mathfrak{p}_x}$ if and only if $\mathfrak{p}_x \supseteq I$, and thus $Z = V(I)$. Now note that $\mathcal{O}_Z$ agrees with the structure sheaf of $\operatorname{Spec} A/I$. Indeed, this follows from the construction of both. We know that $\mathcal{O}_X/\mathcal{I} \cong \widetilde{A/I}$, and we may represent a section of $i^{-1}\mathcal{O}_X/\mathcal{I}$ on an open set $U \subseteq Z$ via (V, s) with s a section of $\mathcal{O}_X/\mathcal{I}$ over V open in X and $U \subseteq V$. However, as the topology on Z is induced by that on X, we may assume $V \cap Z = U$. Then one sees that giving a section of $\widetilde{A/I}$ over V is precisely the same as giving a section of $\mathcal{O}_{\operatorname{Spec} A/I}$ over U, by the construction of both. This shows that $\mathcal{O}_Z = \mathcal{O}_{\operatorname{Spec} A/I}$.

Thus $(Z, \mathcal{O}_Z)$ is a closed subscheme of X.

9. See II Proposition 7.3 of Hartshorne.