NOTES FOR MAKE-UP LECTURE, 17 MAR., 2017

Example 0.1. Let X be a curve of genus 2. Then $\deg K_X = 2$ and $\ell(K_X) = 2$.

Lemma 0.2. *Let X be any projective non-singular curve. If there exists $P, Q \in X$, $P \neq Q$ with $P \sim Q$, then X is isomorphic to $\mathbb{P}^1$.*

Proof. Consider the linear system $|P|$. Since $Q \in |P|$, $\dim |P| \geq 1$, i.e., $\ell(P) \geq 2$. But we have an upper bound $\ell(D) \leq \deg D + 1$, so $\ell(P) = 2$. Now if $Q, R \in X$ are any two points, then $\ell(P - Q - R) = 0$ since $\deg P - Q - R = -1$, so $|P|$ separates points, tanegent vectors, hence induces an embedding of X in $\mathbb{P}^1$, i.e., $X \cong \mathbb{P}^1$. □

We will now show that $|K_X|$ is base-point-free in the situation that the genus is 2. If $|K_X|$ is not base-point-free, then there exists a $P \in X$ such that $\ell(K_X - P) = 2$. Since $\deg K_X - P = 1$, this means there exists $Q, R \in |K_X - P|$, $Q \neq R$, $Q \sim R$, and hence $X \cong \mathbb{P}^1$ by the lemma, contradicting the genus assumption.

So $|K_X|$ induces a degree 2 morphism $X \to \mathbb{P}^1$.

Definition 0.3. A projective non-singular curve X is *hyperelliptic* if there exists a degree 2 morphism $X \to \mathbb{P}^1$.

Thus all degree 2 curves are hyperelliptic.

Theorem 0.4. *Let X be a projective non-singular curve, $g \geq 3$. Then either*

(1) *X is hyperelliptic, or*
(2) *$|K_X|$ induces an embedding $X \to \mathbb{P}^{g-1}$.*

Proof. $|K_X|$ indues an embedding if and only if for all $P, Q \in X$, $\ell(K_X - P - Q) = \ell(K_X) - 2 = g - 2$. In any event,

$$\ell(P + Q) - \ell(K_X - P - Q) = \deg(P + Q) + 1 - g = 3 - g.$$

So $|K_X|$ induces an embedding if and only if $\ell(P + Q) = 1$ for all $P, Q \in X$. So if $|K_X|$ does not induce an embedding, then there exists $P, Q \in X$ such that $\ell(P+Q) \geq 2$. Note that if $\ell(P+Q) \geq 3$, then for $R \in X$, $\ell(P+Q-R) \geq 2$, hence there exists $R_1, R_2 \in |P + Q - R|$ with $R_1 \sim R_2$, so $X \cong \mathbb{P}^1$. Thus $\ell(P + Q) = 2$. Also, $\ell(P + Q - R) = 1$ for all $R \in X$.

Thus we see that if K_X is not very ample, then $|P + Q|$ is base-point-free and induces a degree 2 morphism $X \to \mathbb{P}^1$, so X is hyperelliptic. □

Example 0.5. X a genus 3 curve. If X is not hyperelliptic, then $|K_X|$ induces an embedding in $\mathbb{P}^2$ as a curve of degree 4.

One can show that the genus of a non-singular curve of degree d in $\mathbb{P}^2$ is $(d-1)(d-2)/2$, and in particular, a degree 4 curve in $\mathbb{P}^2$ is genus 3. Further, K_X is the pull-back of a line.

Theorem 0.6 (The Riemann-Hurwitz formula)**.** *Let $f : X \to Y$ be a non-constant morphism between non-singular projective curves, with $K(X)$ separable over $K(Y)$. Then*

$$2 - 2g(X) = (\deg f)(2 - 2g(Y)) - \sum_{p \in X} (e_P - 1).$$

Proof. Proof omitted. Idea: pull-back forms. □

Example 0.7. Let X be a hyperelliptic curve, $Y = \mathbb{P}^1$, $f : X \to Y$ degree 2, so

$$2 - 2g(X) = 4 - \#\{P \in X \mid e_P \neq 1\}.$$

or

$$\#\{P \in X \mid e_P \neq 1\} = 2g(X) + 2.$$

The set here is called *the set of branch points of f*.

For example, consider $X \subseteq \mathbb{P}^2$ an elliptic (i.e., genus one) curve given by $y^2 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$. We define a morphism $X \to \mathbb{P}^1$ by projection from $P_0 = (0 : 1 : 0)$, i.e., $(x : y : z) \mapsto (x : z)$, with $(0 : 1 : 0) \mapsto (1 : 0)$. The branch points are $\lambda_1, \lambda_2, \lambda_3, \infty$. This morphism can be viewed as given by the linear system $|2P_0|$.

Example 0.8. We can construct hyperelliptic curves of any genus as curves in $\mathbb{P}^1 \times \mathbb{P}^1$. A curve X in $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by a *bi-homogeneous* equation $f(u_0, u_1, v_0, v_1) = 0$, where f is homogeneous of degree a in the variables u_0, u_1 and homogeneous of degree b in the variables v_0, v_1. Suppose that $a = 2$. Then consider the projection $f : X \to \mathbb{P}^1$ onto the second factor. Each fibre of this morphism consists of two points (counted with multiplicity), as one just fixes v_0, v_1 and has to solve the (homogeneous) quadratic equation $f(u_0, u_1, v_0, v_1) = 0$. Note that $\alpha u_0^2 + \beta u_0 u_1 + \gamma u_1^2 = 0$ has a double root when the discriminant $\beta^2 - 4\alpha\gamma$ vanishes. However, in this case, the discriminant is a polynomial of degree $2b$ in v_0, v_1. Hence the projection f has $2b$ branch points, and

$$g(X) = (2b - 2)/2 = b - 1.$$