NOTES FOR MAKE-UP LECTURE, FEB. 8, 2017

Recall:

Definition 0.1. n-dimensional projective space over a field k is

$$\mathbb{P}^n = (k^{n+1} \setminus \{0\})/\sim,$$

where $(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$ for $\lambda \in k \setminus \{0\}$.

Note: We often write the equivalence class of $(x_0, \ldots, x_n)$ as $(x_0 : \cdots : x_n)$. (Think ratios!)

Examples 0.2. $\mathbb{P}^0$ is just a point.

$\mathbb{P}^1$. Each point is of the form $(x_0 : x_1)$. If $x_1 \neq 0$, can rescale, and this is equivalent to $(x_0/x_1 : 1)$. We can view this as the point $x_0/x_1 \in \mathbb{A}^1$.

If $x_1 = 0$, then $x_0 \neq 0$ and we can rescale, getting the point $(1 : 0)$. This is called *the point at infinity*, and we can write

$$\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}.$$

For $k = \mathbb{C}$, think Riemann sphere!

$\mathbb{P}^2$: A point is of the form $(x_0 : x_1 : x_2)$. If $x_2 \neq 0$, get $(x_0/x_2 : x_1/x_2 : 1)$ which can be viewed as the point $(x_0/x_2, x_1/x_2) \in \mathbb{A}^1$. If $x_2 = 0$, get $(x_0 : x_1 : 0) \in \mathbb{P}^1$, so

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1 = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \{\infty\}.$$

Here $\mathbb{P}^1$ is the "line at infinity".

In general, $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$, with $\mathbb{P}^{n-1}$ being the hyperplane at infinity.

Can talk about algebraic subsets of $\mathbb{P}^n$. When does $f(x_0, \ldots, x_n) = 0$ make sense?

Definition 0.3. $f \in S := k[x_0, \ldots, x_n]$ is *homogeneous* if every term of f is the same degree, or equivalently,

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n).$$

Here d is the degree of the polynomial f.

Example 0.4. $x_0^3 + x_1x_2^2$ is homogeneous, $d = 3$.
$x_0^3 + x_2^2$ is not homogeneous.

Definition 0.5. If $T \subseteq S$ is a set of homogeneous polyonomials, define

$$Z(T) := \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid f(a_0, \ldots, a_n) = 0 \forall f \in T\}.$$

An ideal $I \subseteq S$ is *homogeneous* if it is generated by homogeneous polynomials. For I homogeneous, we define

$$Z(I) := \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid f(a_0, \ldots, a_n) = 0 \text{ for all } f \in I \text{ homogeneous}\}.$$

A subset of $\mathbb{P}^n$ is *algebraic* if it is of the form $Z(T)$ for some set $T \subseteq S$.

Exercise: Check that the algebraic subsets of $\mathbb{P}^n$ form the closed subsets of a topology on $\mathbb{P}^n$, called the *Zariski topology* on $\mathbb{P}^n$.

Definition 0.6. A *projective variety* is an irreducible closed subset of $\mathbb{P}^n$.

Construction 0.7 (The standard affine open cover of $\mathbb{P}^n$)**.** Define $U_i \subseteq \mathbb{P}^n$ by $U_i := \mathbb{P}^n \setminus Z(x_i)$. This is an open set, and there is a bijection $\varphi_i : U_i \to \mathbb{A}^n$ given by

$$\varphi_i(x_0 : \cdots : x_n) = \left(\frac{x_0}{x_i}, \ldots, \frac{\widehat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i}\right).$$

Here the hat denotes we leave this entry out. Note that U_i carries a topology induced by $\mathbb{P}^n$: open subsets of U_i are those subsets which are also open in $\mathbb{P}^n$.

Proposition 0.8. *φ_i is a homeomorphism.*

Proof. Since φ_i is clearly a bijection, it is enough to show that φ_i identifies closed sets of U_i with closed sets of $\mathbb{A}^n$.

Wlog $i = 0$, $\varphi = \varphi_0$, $U = U_0$. Set $A = k[y_1, \ldots, y_n]$, $S = k[x_0, \ldots, x_n]$, and let $S^h \subseteq S$ denote the set of homogeneous elements of S.

Define maps

$$\alpha : S^h \to A$$
$$\beta : A \to S^h$$

by

$$\alpha(f) = f(1, y_1, \ldots, y_n).$$

If $g \in A$ is of degree e (i.e., the maximal degree of all terms in g), define

$$\beta(g) = x_0^e g(x_1/x_0, \ldots, x_n/x_0).$$

Note. The map β *homgenizes* a polynomial: we introduce a new variable (in this case x_0) and multiply each term of the polynomial g by a sufficiently large power of the variable to make it homogeneous, but in such a way so it is not divisible by the variable. For example,

$$\begin{aligned}\beta(y_2^2 - y_1^3 - y_1 + y_1y_2) &= x_0^3\left(\frac{x_2^2}{x_0^2} - \frac{x_1^3}{x_0^3} - \frac{x_1}{x_0} + \frac{x_1x_2}{x_0^2}\right)\\ &= x_0x_2^2 - x_1^3 - x_1x_2^2 + x_0x_1x_2.\end{aligned}$$

If $Y \subseteq U$ is closed, it is the intersection of U with a closed subset $\overline{Y} \subseteq \mathbb{P}^n$, which can be assumed to be the closure of Y in $\mathbb{P}^n$. So $\overline{Y} = Z(T)$ for some $T \subseteq S^h$. Let $T' = \alpha(T)$. Then $\varphi(Y) = Z(\alpha(T))$. Indeed, for $(a_0, \dots, a_n) \in U$,

$$\begin{aligned}f(a_0, \dots, a_n) = 0 &\Leftrightarrow f(1, a_1/a_0, \dots, a_n/a_0) = 0\\ &\Leftrightarrow \alpha(f)(a_1/a_0, \dots, a_n/a_0) = 0\\ &\Leftrightarrow \alpha(f)(\varphi(a_0, \dots, a_n)) = 0.\end{aligned}$$

Conversely, for $W \subseteq \mathbb{A}^n$ closed, $W = Z(T')$ for some $T' \subseteq A$, and $\varphi^{-1}(W) = Z(\beta(T')) \cap U$. Indeed,

$$\begin{aligned}g(b_1, \dots, b_n) = 0 &\Leftrightarrow \beta(g)(1, b_1, \dots, b_n) = 0\\ &\Leftrightarrow \beta(g)(\varphi^{-1}(b_1, \dots, b_n)) = 0.\end{aligned}$$

□

Example 0.9. Consider map $f : \mathbb{P}^1 \to \mathbb{P}^3$,

$$f(u : t) = (u^3 : u^2t : ut^2 : t^3).$$

The image of f is called the *twisted cubic.*

How do we show image is an algebraic subset of $\mathbb{P}^3$?

Consider map

$$\varphi : k[x_0, x_1, x_2, x_3] \to k[u, t]$$

defined by

$$\begin{aligned}x_0 &\mapsto u^3\\ x_1 &\mapsto u^2t\\ x_2 &\mapsto ut^2\\ x_3 &\mapsto t^3\end{aligned}$$

If $I = \ker \varphi$, then for any homogeneous $g \in I$, g vanishes on the image of f. Thus the image of f is contained in $Z(I)$.

Conversely, note $x_0x_3 - x_1x_2, x_1^2 - x_0x_2, x_2^2 - x_1x_3 \in I$. Suppose $p \in Z(I)$, $p = (a_0 : \cdots : a_3)$. Analyze four cases. If $a_0 \neq 0$, can assume $a_0 = 1$. Then $a_3 = a_1a_2$, $a_2 = a_1^2$, so $p = (1, a_1, a_1^2, a_1^3) = f(1, a_1)$.

Similar arguments work for when $a_i \neq 0$, $i = 1, 2, 3$.