Part III Algebraic Geometry 2020 **Mark Gross**

Introductory Reading

[Has] B. Hassett, Introduction to Algebraic Geometry, Cambridge University Press, 2007.

[R] M. Reid, Undergraduate Algebraic Geometry, Cambridge University Press (1988).

Standard References for Commutative Algebra

[AM] M. Atiyah and I. MacDonald, Introduction to Commutative Algebra, Addison–Wesley (1969)

[Ma1] H. Matsumura, Commutative Algebra, 2nd edition. The Benjamin/Cummings Publishing Company, 1980.

[Ma2] H. Matsumura, Commutative Ring Theory, Cambridge University Press, 1986.

Main references for course

[Har] R. Hartshorne, Algebraic Geometry, Springer (1977).

[GW] Görtz and Wedhorn, Algebraic Geometry I, Schemes with examples and exercises, Vieweg+Teubner, 2010.

[V] R. Vakil, The Rising Sea: Foundations of Algebraic Geometry, the latest version available at http://math.stanford.edu/∼vakil/216blog/

Plan of course

§0 Brief review of classical Algebraic Geometry and motivation for scheme theory
§1 Sheaf Theory
§2 Definition of schemes, basic properties and examples
§3 Locally free and coherent modules
§4 Sheaf cohomology
§5 Differentials and Riemann-Roch for curves

§0. Preliminaries on classical Algebraic Geometry and Commutative Algebra

In this section, I shall make explicit basic concepts and results that I am assuming from elsewhere; all this material (should have) appeared in Part II Algebraic Geometry. For more details, see [Has] or [R]. The first section on Sheaf Theory will take several lectures and will only incidentally need any algebraic geometry, and so the reader has some time to familiarise himself/herself with the material in §0.

A little classical algebraic geometry.
(Throughout this discussion, we take the base field k to be **algebraically closed**.)

Affine varieties: An *affine variety* $V \subseteq \mathbf{A}^n(k)$ (where, once one has chosen coordinates, $\mathbf{A}^n(k) = k^n$) is given by the vanishing of polynomials $f_1, \dots, f_r \in k[X_1, \dots, X_n]$.
If $I = \langle f_1, \dots, f_r \rangle \lhd k[X_1, \dots, X_n]$ is any ideal, we set

$$V = V(I) := \{z \in \mathbf{A}^n \ : \ f(z) = 0 \ \forall f \in I\}.$$

Projective varieties: First set $\mathbf{P}^n(k) := (k^{n+1} \setminus \{\mathbf{0}\})/k^*$ with *homogeneous coordinates* $(x_0 : x_1 : \dots : x_n)$. A *projective variety* $V \subseteq \mathbf{P}^n$ is given by the vanishing of homogeneous polynomials $F_1, \dots, F_r \in k[X_0, X_1, \dots, X_n]$. If I is the ideal generated by the F_i (a *homogeneous* ideal, i.e. if $F \in I$, then so are all its homogeneous parts), we set

$$V = V(I) := \{z \in \mathbf{P}^n \ : \ F(z) = 0 \ \forall \text{ homogeneous } F \in I\}.$$

Coordinate ring of an affine variety.

If $V = V(I) \subseteq \mathbf{A}^n$, set

$$I(V) := \{f \in k[X_1, \dots, X_n] \ : \ f(x) = 0 \ \forall x \in V\}.$$

Observe: $V(I(V)) = V$ (tautology) and $I(V(I)) \supseteq \sqrt{I}$ (obvious). Recall that the radical $\sqrt{I}$ of the ideal I is defined by $f \in \sqrt{I} \iff \exists m > 0$ s.t. $f^m \in I$.

Hilbert's Nullstellensatz (note $k = \bar{k}$): $I(V(I)) = \sqrt{I}$. ([R] §3, [AM] pp 82-3).

Coordinate ring: $k[V] := k[X_1, \dots, X_n]/I(V)$. This may be regarded as the ring of polynomial functions on V, and it is a finitely generated reduced k-algebra. Recall that a k-algebra is a commutative ring containing k as a subring; it is finitely generated if it is the quotient of a polynomial ring over k, and *reduced* if $a^m = 0 \Rightarrow a = 0$.

Given an affine subvariety $W \subseteq V$, have $I(W) \supseteq I(V)$ defining an ideal of $k[V]$, also denoted $I(W) \lhd k[V]$.

Corollary of 0-satz: If $\mathbf{m}$ is a maximal ideal of $k[V]$, then $\mathbf{m} = \mathbf{m}_P$ for some $P \in V$, where $\mathbf{m}_P$ is the maximal ideal $\{f \in k[V] \ : \ f(P) = 0\}$.

Proof. 0-satz implies $I(V(\mathbf{m})) = \sqrt{\mathbf{m}} = \mathbf{m} \neq k[V]$. So $V(\mathbf{m}) \neq \emptyset$, since otherwise $I(V(\mathbf{m})) = k[V]$. Choose $P \in V(\mathbf{m})$; then $\mathbf{m}_p \supseteq \mathbf{m}$. Since $\mathbf{m}$ maximal, this implies $\mathbf{m}_P = \mathbf{m}$.

Observe that $\{P\} = V(\mathbf{m}_P) = V(\mathbf{m})$, and so there exists a natural bijection

$$\{\text{points of affine variety } V\} \longleftrightarrow \{\text{maximal ideals of } k[V]\} \qquad (\dagger)$$

Definition. A variety W is *irreducible* if there do not exist proper subvarieties W_1, W_2 of W with $W = W_1 \cup W_2$.

Lemma 0.1. A subvariety W of an affine variety V is irreducible $\iff$ $\mathcal{P} = I(W)$ is prime, i.e. $\iff$ $k[W]$ is an ID (integral domain).

Proof. ($\Rightarrow$) If $I(W)$ not prime, there exist $f, g \notin I(W)$ such that $fg \in I(W)$. Set $W_1 := V(f) \cap W$ and $W_2 := V(g) \cap W$; then W_1, W_2 are proper subvarieties with $W = W_1 \cup W_2$, i.e. W not irreducible.

($\Leftarrow$) If W_1, W_2 are proper subvarieties with $W = W_1 \cup W_2$, choose $f \in I(W_1) \setminus I(W)$ and $g \in I(W_2) \setminus I(W)$; then $fg \in I(W)$, i.e. $I(W)$ not prime.

For a projective variety $V \subseteq \mathbf{P}^n$, we let $I(V) \lhd k[X_0, X_1, \ldots, X_n]$ be the *homogeneous ideal* of V, by definition generated by the homogeneous polynomials vanishing on V.

Exercise. Show that a projective variety V is irreducible $\iff$ $I(V)$ is prime.
(($\Leftarrow$) as in (0.1), ($\Rightarrow$) by considering homogeneous parts of polynomials.)

Generalizing ($\dagger$), for V an affine variety, we have a bijection given by $W \mapsto I(W)$,

$$\{\text{irreducible subvarieties } W \text{ of an affine variety } V\} \longleftrightarrow \{\text{prime ideals of } k[V]\}.$$

Proof. Given a prime ideal $\mathcal{P} \lhd k[V]$, the Nullstellensatz implies $I(V(\mathcal{P})) = \sqrt{\mathcal{P}} = \mathcal{P}$ in $k[V]$, so there is an inverse map.

Projective Nullstellensatz. Suppose I is a homogeneous ideal in $k[X_0, X_1, \ldots, X_n]$ and $V = V(I) \subseteq \mathbf{P}^n$. The Projective Nullstellensatz ([R] p82) says:
If $\sqrt{I} \neq \langle X_0, X_1, \ldots, X_n \rangle$ (the *irrelevant* ideal), then $I(V) = \sqrt{I}$.

Proof. An easy deduction from the Affine Nullstellensatz, noting that I also defines an affine variety in $\mathbf{A}^{n+1}$, the *affine cone* on the projective variety $V \subseteq \mathbf{P}^n$.

Decomposition of variety into irreducible components.

For V an affine or projective variety, there is a decomposition $V = V_1 \cup \ldots \cup V_N$ with the V_i irreducible subvarieties and the decomposition is essentially unique.

Proof. Suppose V is affine (similar argument for V projective): If there does not exist such a finite decomposition in the above form, then there exists a strictly decreasing sequence of subvarieties

$$V = V_0 \supset V_1 \supset V_2 \supset \ldots.$$

(If $V = W \cup W'$, then at least one of W, W' has no such decomposition and let this be V_1; continue in same way using Countable Axiom of Choice to obtain sequence.)

Hence in $k[V]$, $0 = I(V_0) \subseteq I(V_1) \subseteq \ldots.$ Hilbert's Basis Theorem implies that there exists N such that $I(V_{N+r}) = I(V_N)$ for all $r \geq 0$. Hence $V_{N+r} = V(I(V_{N+r})) = V(I(V_N)) = V_N$ for all $r \geq 0$, a contradiction.

An easy "topological" argument ([R] Exercise 3.8, [W]) with the Zariski topology (see below) shows that the decomposition is essentially unique.

Zariski topology. Let V be a variety (affine or projective), then the *Zariski topology* is the topology on V whose closed sets are the subvarieties. This is the underlying topology for this course

We check this is a topology. Wlog take V affine. Clearly V and $\emptyset$ are closed. Observe that for ideals $(I_\alpha)_{\alpha \in A}$ of $k[V]$, we have $V(\sum_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$ is closed. Finally we claim for ideals I, J of $k[V]$ that $V(IJ) = V(I) \cup V(J)$ $(= V(I \cap J))$ is closed.

Proof. Clearly $V(IJ) \supseteq V(I \cap J) \supseteq V(I) \cup V(J)$. Suppose however there exists a point $P \in V(IJ) \setminus (V(I) \cup V(J))$: we can choose $f \in I$ such that $f(P) \neq 0$ and $g \in J$ such that $g(P) \neq 0$. Then $fg \in IJ$ with non-zero value at P, a contradiction.

Note that V being irreducible as a topological space corresponds to the previous definition. Also, we have a well-defined concept of connectedness.

When V is affine, we have a basis of open sets of the form $D(f)$ for $f \in k[V]$, where $D(f) := \{x \in V \ : \ f(x) \neq 0\}$; any *open* set is of the form $V \setminus V(f_1, \ldots, f_r) = \bigcup_{i=1}^r D(f_i)$. If $V = \mathbf{A}^1$, get *cofinite* topology; in fact Zariski topology is only Hausdorff for a finite set of points. For V projective, we have a basis of open sets of the form $D(F) = V \setminus V(F)$, for F a homogeneous polynomial.

Exercise. The Zariski topology is compact in the usual sense (called precompact in some terminology since it is not Hausdorff), i.e. any open cover of V has a finite subcover.

Function fields of irreducible varieties

If V is an *irreducible* affine variety, then the *field of rational functions* or the *function field* $k(V) := \text{fof}\, k[V]$. Here $k[V]$ is an integral domain and fof denotes the field of fractions. In fact, we define the *dimension* of V by $\dim V := \text{tr deg}_k\, k(V)$.

For $V \subseteq \mathbf{P}^n$ an irreducible projective variety, we define

$$k(V) := \{F/G \ : \ F, G \text{ homogeneous polynomials of the same degree, } G \notin I(V)\}/\sim$$

where the zero polynomial has any degree and where $F_1/G_1 \sim F_2/G_2 \iff F_1G_2 - F_2G_1 \in I(V)$. Need V irreducible here, i.e. $I(V)$ prime, to show that $\sim$ is transitive, and hence an equivalence relation.

If $V \subseteq \mathbf{P}^n$ an irreducible projective variety and U a non-empty affine piece of V (say $U = V \cap \{X_0 \neq 0\}$), then U is an affine variety, $U \subseteq \mathbf{A}^n$ with affine coordinates $x_i = X_i/X_0$ for $i = 1, \ldots, n$, the equations for U coming from those for V by "putting $X_0 = 1$". It is an easy check now that U is irreducible and $k(V) \cong k(U)$, the isomorphism being given by "putting $X_0 = 1$".

We say that $h \in k(V)$ is *regular* at $P \in V$ if it can be written as a quotient f/g with $f, g \in k[V], g(P) \neq 0$ (affine case), or F/G with F, G homogeneous polynomials of the same degree, $G(P) \neq 0$ (projective case).

Define $\mathcal{O}_{V,P} := \{h \in k(V) \ : \ h \text{ regular at } P\}$, the *local ring of* V *at* P, with maximal ideal $\mathbf{m}_{V,P} := \{h \in \mathcal{O}_{V,P} \ : \ h(P) = 0\}$, the kernel of the *evaluation map* $\mathcal{O}_{V,P} \to k$ given by evaluation at P. $\mathcal{O}_{V,P}$ is a *local ring,* i.e. $\mathbf{m}_{V,P}$ is the unique maximal ideal. Since $\mathcal{O}_{V,P} \setminus \mathbf{m}_{V,P}$ consists of units of $\mathcal{O}_{V,P}$ and any proper ideal consists of non-units, any proper ideal is contained in $\mathbf{m}_{V,P}$, and hence $\mathbf{m}_{V,P}$ is the unique maximal ideal.

Morphisms of affine varieties

For $V \subseteq \mathbf{A}^n$, $W \subseteq \mathbf{A}^m$, a morphism $\phi : V \to W$ is a map given by elements $\phi_1, \ldots, \phi_m \in k[V]$. This yields a k-algebra homomorphism $\phi^* : k[W] \to k[V]$ (where $\phi^*(f) = f \circ \phi$; so if y_j a coordinate function on W induced from polynomial Y_j, we have $\phi^*(y_j) = \phi_j$). Conversely, given a k-algebra homomorphism $\alpha : k[W] \to k[V]$, we define a morphism $\alpha^* = \psi : V \to W$ given by elements $\alpha(y_1), \ldots, \alpha(y_m) \in k[V]$. Note that $\psi(P)$ is in W, since for all $g \in I(W)$, $g(\psi(P)) = g(\alpha(y_1), \ldots, \alpha(y_m))(P) = (\alpha(g(y_1, \ldots, y_m)))(P) = 0$ since $g(y_1, \ldots, y_m) = 0$ in $k[W]$.

Observe: For $\phi : V \to W$, we have $\phi^{**} = \phi$; for $\alpha : k[W] \to k[V]$, we have $\alpha^{**} = \alpha$. For $\psi : U \to V$ also a morphism of affine varieties, we have $\phi\psi$ a morphism $U \to W$ with $(\phi\psi)^* = \psi^*\phi^*$. For $\beta : k[V] \to k[U]$ a morphism of k-algebras, we have $(\beta\alpha)^* = \alpha^*\beta^*$.

We deduce that affine varieties V, W are *isomorphic* (i.e. there is an invertible morphism between them) $V \cong W \iff k[W] \cong k[V]$ as k-algebras. Recall: the k-algebras which occur as coordinate rings are the finitely generated reduced k-algebras. So formally, there is an equivalence of categories between the category of affine varieties over k and their morphisms, and the opposite of the category of finitely generated reduced k-algebras and their morphisms, i.e. there is a contravariant equivalence between the category of affine varieties and the category of finitely generated reduced k-algebras.

Thus affine algebraic geometry over k is a branch of commutative algebra. Commutative Algebra may be interpreted as affine algebraic geometry once one has generalized varieties to *schemes*.

For (irreducible) *affine* varieties, we can reconstruct the variety (up to isomorphism) from its ring of everywhere regular rational functions by (0.2) below; for irreducible projective varieties, the only everywhere regular rational functions are the constants (see Corollary 2 to Proposition 2.2).

Lemma 0.2. For V an irreducible affine variety,

$$\{f \in k(V) \ : \ f \text{ regular everywhere}\} = k[V].$$

Proof. Exercise.

A little Commutative Algebra

Let A be a commutative ring (with a 1).

Definition. A module M over A is *finitely generated* if $\exists n > 0$ and $x_1, \dots, x_n \in M$ such that $M = Ax_1 + \cdots + Ax_n$ ($\iff$ M is a quotient of the free module A^n).

Nakayama's lemma ([AM] p21)
If M is a finitely generated module over a local ring $(A, \mathbf{m})$, where $\mathbf{m}$ is the unique maximal ideal of A, such that $M = \mathbf{m}M$, then $M = 0$.

A useful corollary of this is with above notation and $N \subseteq M$ a submodule with $M = \mathbf{m}M + N$, then $M = N$ (apply Nakayama to quotient module M/N).

Rings and modules of fractions. Let A be a commutative ring, $S \subseteq A$ a *multiplicative subset* (i.e. $1 \in S$ and $s, t \in S \Rightarrow st \in S$). We can define an equivalence relation $\sim$ on $A \times S$ by $(a, s) \sim (a', s') \iff t(as' - a's) = 0$ for some $t \in S$ (easy check that $\sim$ is an equivalence relation). Let a/s denote the equivalence class of (a, s) and $S^{-1}A$ the set of such classes a/s. Define addition and multiplication in the obvious way. Then $S^{-1}A$ is a commutative ring and there exists a natural ring homomorphism $\phi : A \to S^{-1}A$, namely $\phi(a) = a/1$. $S^{-1}A$ is called the *ring of fractions* of A w.r.t. S.

There is a universal property: If $g : A \to B$ is a homorphism of rings with $g(S) \subseteq U(B)$ (units of B), then $\exists!\ g' : S^{-1}A \to B$ with $g'\phi = g$ (namely $g'(a/s) = g(a)g(s)^{-1} \in B$).

$S^{-1}A$ has a 1 (= 1/1) and a zero (= 0/1). Then $a/s = 0 \iff ta = 0$ for some $t \in S$; hence $S^{-1}A = 0 \iff 1/1 = 0/1 \iff 0 \in S$.

The map $A \to S^{-1}A$ is an isomorphism $\iff$ $S \subseteq U(A)$ (for $(\Leftarrow)$, take $B = A$ in universal property).

Let $T \subset A$ be the set of non divisors of zero, a multiplicative subset. Set $T^{-1}A = \mathrm{tot}(A)$, the *total ring of fractions* — we have an injection $A \hookrightarrow \mathrm{tot}(A)$. If A is an integral domain (ID), then $\mathrm{tot}(A) = \mathrm{fof}(A)$ (taking $T = A \setminus \{0\}$). For a *reducible* affine variety V, we should replace the function field $k(V)$ by the *ring* $\mathrm{Rat}(V) := \mathrm{tot}(k[V])$ of rational functions on V.

Relevant examples

(1) If $f \in A$, let $f^{\mathbf{N}} = \{1, f, f^2, \dots\} = S$. Write A_f for $S^{-1}A$ in this case.

(2) If $\mathcal{P}$ is a prime ideal of A, then $S = A \backslash \mathcal{P}$ is a multiplicative subset. Write $A_{\mathcal{P}}$ for $S^{-1}A$,

called the *localisation* of A at $\mathcal{P}$, a local ring with unique maximal ideal $\mathcal{P}A_{\mathcal{P}}$ consisting of elements a/s with $a \in \mathcal{P}$, $s \notin \mathcal{P}$ (all the other elements of $A_{\mathcal{P}}$ are units).

If now M is an A-module, $S \subseteq A$ a multiplicative subset, the *module of fractions* $S^{-1}M$ (both an A-module and an $S^{-1}A$-module) is defined analogously, with $m/s = m'/s' \iff t(s'm - sm') = 0$ for some $t \in S$. The $S^{-1}A$-module structure is defined via $(a/s).(m/t) = (am)/(st)$.

Tensor products

Definition, The tensor product $M \otimes_A N$ of A-modules M and N is an A-module equipped with an A-bilinear map $g : M \times N \to M \otimes_A N$ with the following universal property:

Given any A-bilinear map $f : M \times N \to P$, $\exists!$ morphism of A-modules $h : M \otimes_A N \to P$ which factorizes $f = hg$.

$M \otimes_A N$ is defined up to isomorphism by this property (easy application of universal property). The existence of such a module is straightforward and unenlightening (see [AM] p 24) — take the free module F over A on the set $M \times N$ and quotient out by the appropriate submodule of bilinear relations. We omit the subscript A where no confusion would result in doing so. We denote by $x \otimes y$ the image of (x, y) in $M \otimes_A N$.

Elementary properties (all proved from universal property, [AM] p 26)

If M, N, P are A-modules, there exist isomorphisms of A-modules

- $M \otimes N \cong N \otimes M$, where $x \otimes y \mapsto y \otimes x$.
- $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$, where $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$.
- $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$, where $(x, y) \otimes z \mapsto (x \otimes z, y \otimes z)$.
- $A \otimes M \cong M$, where $a \otimes x \mapsto ax$.

Change of ring: Given a morphism of rings $f : A \to B$ (NB $f(1) = 1$), we call B an A-algebra – this generalises previous concept of k-algebras. Given an A-algebra structure on B, $f : A \to B$, and an A-module M, set $M_B := B \otimes_A M$; this is also a B-modules in an obvious way with B acting on the first factor.

Proposition 0.2. Let M be an A-module.

(a) If $I \lhd A$ and $B = A/I$, then $B \otimes_A M \cong M/IM$.

(b) If $S \subseteq A$ is a multiplicative subset and $B = S^{-1}A$, then $B \otimes_A M \cong S^{-1}M$ (this is therefore an alternative definition).

Proof. (a) The obvious bilinear map $(A/I) \times M \to M/IM$ induces (using universal property) a morphism of A-modules $(A/I) \otimes_A M \to M/IM$, where for any $a \in A$, $x \in M$, we have $\bar{a} \otimes x \mapsto \overline{ax}$. The inverse morphism $M/IM \to (A/I) \otimes_A M$ is given by $\bar{x} \mapsto 1 \otimes x$ (check well-defined).

(b) Use universal properties of both S^{-1} and $\otimes_A$ — see [AM] p 40.

Proposition 0.3. If M, N are A-modules, $I \lhd A$, S a multiplicative subset of A, then

(a) $(A/I) \otimes_A (M \otimes_A N) \cong (M/IM) \otimes_{A/I} (N/IN)$,

(b) $S^{-1}(M \otimes_A N) \cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N$.

Proof. Exercise.

For instance, if $\mathcal{P}$ a prime ideal of A, then $(M \otimes_A N)_{\mathcal{P}} \cong M_{\mathcal{P}} \otimes_{A_{\mathcal{P}}} N_{\mathcal{P}}$ (where we define $M_{\mathcal{P}} = (A \setminus \mathcal{P})^{-1}M$, etc.).

***R*-algebras.** Given a commutative ring R and R-algebras $\theta_1 : R \to A$, $\theta_2 : R \to B$, a morphism $A \to B$ of R-algebras is given by morphism of rings $f : A \to B$ such that $f\theta_1 = \theta_2$. Given R-algebras A and B, the tensor product $A \otimes_R B$ has the structure of an R-algebra:

- Multiplication given by $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$, and extend linearly.
- The ring homomorphism $R \to A \otimes_R B$ given by $r \mapsto \theta_1(r) \otimes 1 = 1 \otimes \theta_2(r)$.

Also have R-algebra morphisms $\alpha : A \to A \otimes_R B$ and $\beta : B \to A \otimes_R B$ given by $a \mapsto a \otimes 1$, respectively $b \mapsto 1 \otimes b$. These satisfy a *universal property* that, given any R-algebra morphisms $\alpha' : A \to C$ and $\beta' : B \to C$, $\exists!$ R-algebra morphism $\phi : A \otimes_R B \to C$ such that $\alpha' = \phi\alpha$ and $\beta' = \phi\beta$. Moreover $A \otimes_R B$ is determined (up to isomorphism) by this universal property (check).

Using this, we can deduce for R-algebras A, B, C that $A \otimes_R (B \otimes_R C) \cong (A \otimes_R B) \otimes_R C$ etc. are naturally isomorphic as R-algebras (rather than just R-modules).