Part III Topics in Algebraic Geometry

Example Sheet II, 2019.

Note: If you would like to receive feedback, please turn in solutions to Questions 2, 7, and 9 by Thursday, November 14th. at noon, at which time solutions will be posted. You may turn in your work either in lecture or in my CMS pigeon hole. The second examples class will be on Thursday, November 14th, at 3:30 pm in MR5.

1. We check the details of the construction of Proj in lecture. Recall, in analogy with Spec A, an affine scheme, we can define a *projective scheme*. Let $S = \bigoplus_{d=0}^{\infty} S_d$ be a graded ring. We will define $\operatorname{Proj} S$.

   i) We write $S^+ = \bigoplus_{d=1}^{\infty} S_d$, the *irrelevant ideal*. Define $\operatorname{Proj} S$ to be the set of all homogeneous prime ideals of S not containing the irrelevant ideal. If $I \subseteq S$ is a homogeneous ideal, let $V(I)$ denote the set of all primes in $\operatorname{Proj} S$ containing I. Show these form the closed sets of a topology on $\operatorname{Proj} S$.

   ii) We define a sheaf $\mathcal{O}$ on $\operatorname{Proj} S$. For $\mathfrak{p} \in \operatorname{Proj} S$, let $S_{(\mathfrak{p})}$ be the set of elements of the localization $S_{\mathfrak{p}}$ which are homogeneous of degree zero (i.e., a ratio of two elements of S of the same degree). Define for $U \subseteq \operatorname{Proj} S$ open the ring $\mathcal{O}(U)$ to be the set of functions

   $$s : U \to \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})}$$

   such that $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ and every point $\mathfrak{p} \in U$ has an open neighbourhood V for which there exists $f, g \in S$ homogeneous of the same degree, $g \notin \mathfrak{q}$ for all $\mathfrak{q} \in V$, such that $s(\mathfrak{q}) = f/g$ for $\mathfrak{q} \in V$. Then show:

   a) The stalk of $\mathcal{O}$ at $\mathfrak{q}$ is $S_{(\mathfrak{q})}$.

   b) For any homogeneous $f \in S_+$, let $D_+(f)$ be the set of primes of $\operatorname{Proj} S$ not containing f. Show the sets $D_+(f)$ cover $\operatorname{Proj} S$, and for each such open set, there is an isomorphism of locally ringed spaces

   $$(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \operatorname{Spec} S_{(f)}.$$

   Here $S_{(f)}$ denotes the subring of elements of degree 0 in the localization S_f. [Hint: This is a bit tricky. It is easy to define a map $\psi : D_+(f) \to \operatorname{Spec} S_{(f)}$. However, constructing its inverse θ is not so easy. Given $\mathfrak{q} \in \operatorname{Spec} S_{(f)}$, set

   $$\mathfrak{q}_n := \{s \in S_n \mid s^{\deg f}/f^n \in \mathfrak{q}\}.$$

   Show that $\mathfrak{q}_n$ is closed under addition and that $\theta(\mathfrak{q}) = \bigoplus_n \mathfrak{q}_n$ is a homogenoeous prime ideal in $D_+(f)$.]

   c) $\operatorname{Proj} S$ is a scheme.

   d) Show that if k is an algebraically closed field, then the set of closed points (i.e., points x such that the closure of $\{x\}$ is $\{x\}$) of $\operatorname{Proj} k[x_0, \ldots, x_n]$ are in one-to-one correspondence with points of $(k^{n+1} \setminus \{0\})/k^*$, with the usual action of k^* given by scalar multiplication. Show that if $I \subseteq k[x_0, \ldots, x_n]$ is a homogeneous ideal, then the closed points of $\operatorname{Proj} k[x_0, \ldots, x_n]/I$ are in one-to-one correspondence with equivalence classes of points $(a_0, \ldots, a_n) \in (k^{n+1} \setminus \{0\})/k^*$ such that $f(a_0, \ldots, a_n) = 0$ for all $f \in I$ homogeneous.

2. Let X be a scheme, with open affine subsets $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$. Show that $U \cap V$ can be covered by open affine subschemes $\{U_i\}$ such that there are elements $f_i \in A$, $g_i \in B$ with $U_i = D(f_i) \subset U$ and $U_i = D(g_i) \subseteq V$.

We will now define a number of properties of schemes and morphisms of schemes. This material can be found as a mixture of the text and the exercises of Chapter II, §3 of Hartshorne. Consult that text if you get stuck!

3. We say a scheme X is *irreducible* if it is irreducible as a topological space, i.e., whenever $X = X_1 \cup X_2$ with X_1, X_2 closed subsets, then either $X_1 = X$ or $X_2 = X$.

   We say a scheme X is *reduced* if for every $U \subseteq X$ open, $\mathcal{O}_X(U)$ has no nilpotents.

   We say a scheme X is *integral* if for every $U \subseteq X$ open, $\mathcal{O}_X(U)$ is an integral domain.

   Show that a scheme is integral if and only if it is reduced and irreducible.

4. We say a scheme is *locally Noetherian* if it can be covered by affine open subsets $\operatorname{Spec} A_i$ with A_i a Noetherian ring. We say a scheme is *Noetherian* if it can be covered by a *finite* number of open affine subsets $\operatorname{Spec} A_i$ with A_i Noetherian.

   Show that a scheme X is locally Noetherian if and only if for every open affine subset $U = \operatorname{Spec} A$, A is a Noetherian ring. [Hint: This is II Prop. 3.2 in Hartshorne. Do have a go at this before you look at his proof. At least try to reduce to the following statement before you peek: given a ring A and a finite collection of elements $f_i \in A$ which generate the unit ideal, suppose A_{f_i} is Noetherian for each i. Then A is Noetherian.]

5. A morphism $f : X \to Y$ is *locally of finite type* if there exists a covering Y by open affine subsets $V_i = \operatorname{Spec} B_i$, such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra.

   The morphism is *of finite type* if the cover of $f^{-1}(V_i)$ above can be taken to be finite.

   Show that a morphism $f : X \to Y$ is locally of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

6. For each of the properties defined above of schemes or morphisms, given an example of a scheme or morphism which violates that property. Give an example of a morphism which is locally of finite type but not of finite type.

   Remark. In the language above, we defined a variety as an integral scheme of finite type over $\operatorname{Spec} k$, for k an algebraically closed field.

7. Let X be an integral scheme. Show there is a unique point η such that the closure of $\{\eta\}$ is X; this is called the *generic point* of X. Show that the stalk of $\mathcal{O}_X$ at η is a field, called the *function field* of X, denoted by $K(X)$. Show that if $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is the field of fractions of A.

8. *Normalization.* A scheme is *normal* if all its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of X, let $\tilde{A}$ be the integral closure of A in its quotient field, and let $\tilde{U} = \operatorname{Spec} \tilde{A}$. Show that one can glue the schemes $\tilde{U}$ to obtain a normal integral scheme $\tilde{X}$, called the *normalization* of X. Show that there is a morphism $\tilde{X} \to X$ having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\tilde{X}$. [A morphism $f : Z \to X$ is *dominant* if $f(Z)$ is a dense subset of X.]

9. Describe the fibres over all points of the target space for the following morphisms between affine schemes. In each case, the corresponding homomorphism of rings is the obvious one. Here k denotes a field. Which fibres are irreducible or reduced?

   1. $\operatorname{Spec} k[T,U]/(TU-1) \to \operatorname{Spec} k[T]$.
   2. $\operatorname{Spec} k[T,U]/(T^2-U^2) \to \operatorname{Spec} k[T]$.
   3. $\operatorname{Spec} k[T,U,V,W]/((U+T)W, (U+T)(U^3+U^2+UV^2-V^2))) \to \operatorname{Spec} k[T]$.
   4. $\operatorname{Spec} \mathbb{Z}[T] \to \operatorname{Spec} \mathbb{Z}$.
   5. $\operatorname{Spec} \mathbb{Z}[T]/(T^2+1) \to \operatorname{Spec} \mathbb{Z}$. [Number theorists: what does the calculation you just did mean from the point of view of algebraic number theory?]
   6. $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Z}$.

10. We say a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

   is *Cartesian* if the induced morphism $A \to B \times_D C$ is an isomorphism. Show the following diagrams are Cartesian in any category:

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ X \times_T Y & \longrightarrow & S \times_T S \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle \Gamma_f}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X \times_S Y & \longrightarrow & Y \times_S Y \end{array}$$

   In the first diagram one is given morphisms $X, Y \to S \to T$, and the morphism Δ is induced by the universal property of $S \times_T S$ using the identity $S \to S$ twice. In the second diagram, we assume given X, Y objects over S and a morphism $f : X \to Y$ over S. Then Δ is defined as before and Γ_f is the morphism induced by the identity $X \to X$ and $f : X \to Y$.