Donaldson–Thomas invariants + Bridgeland stability conditions.

A Calabi–Yau variety is a non-singular projective variety X with $\omega_X \cong \mathcal{O}_X$. Alternatively, X

↑ canonical bundle, also think of $K_X = 0$.

carries a nowhere vanishing holomorphic n-form, where $n = \dim X$.

Examples: A degree $n+1$ hypersurface in $\mathbb{P}^n$ is a Calabi–Yau variety (e.g., quintic hypersurface in $\mathbb{P}^4$).

We are interested in the case of 3-dimensional Calabi–Yau's.

DT invariants count vector bundles on CY 3-folds.

Requirements:

(1) A good notion of a moduli space of vector bundles ⇝ Restrict attention to stable vector bundles.

(2) Some way of "counting" points in the moduli space.

These moduli spaces are expected to have dimension 0.

⇝ virtual fundamental class

See Reference 5

R. Thomas "A holomorphic Casson invariant ..."

We can replace counting vector bundles with counting "stable" objects in the derived category of coherent sheaves on X.

(Hartshorne, Residues + Duality.
Thomas, Derived categories for the working mathematician.)

This requires a new notion of stability.

Tom Bridgeland, Stability conditions on triangulated categories.

P. Aspinwall, Dirichlet branes and Mirror Symmetry.

Kontsevich + Soibelman, "Stability structures, Motivic DT invariants and cluster transformations"

Pandharipande + Thomas "13 1/2 ways of counting curves."

Mirror Symmetry essay.

<u>Def</u>: A <u>Calabi-Yau variety</u> is a projective non-singular variety with trivial canonical bundle $\omega_X \cong \mathcal{O}_X$.

i.e., X has a nowhere vanishing holomorphic $n = \dim X$ - form.

<u>E.g.</u> A hypersurface of degree $n+1$ in $\mathbb{CP}^n$

(cubic curve in $\mathbb{P}^2$
quartic surface in $\mathbb{P}^3$
<u>quintic hypersurface</u> in $\mathbb{P}^4$)

The quintic 3-fold.

String theory: 10-dim'l universe

$\mathbb{R}^{1,3} \times X$

6-real dim'l, very small compact manifold

X should be a C-Y 3-fold

1989 Lots of examples of pairs of Calabi-Yau 3-folds X, $\check{X}$,

$$\chi_{top}(X) = -\chi_{top}(\check{X})$$

$$\dim H^{p,q}(X) = \dim H^{p,3-q}(\check{X})$$

<u>Explicit example</u>.

X a quintic 3-fold

$$x_0^5 + \cdots + x_4^5 = 0$$

There is an action of $\underset{G}{\mathbb{Z}_5^3} \subseteq \dfrac{\mathbb{Z}_5^5}{\mathbb{Z}_5}$

$$G = \frac{\{(a_0, \ldots, a_4) \mid a_i \in \mathbb{Z}_5, \sum a_i = 0\}}{\{(a, \ldots, a) \mid a \in \mathbb{Z}_5\}}$$

G acts on $\mathbb{P}^4$ via $(a_0, \ldots, a_4)$ acts by $(x_0, \ldots, x_4) \mapsto (\zeta^{a_0} x_0, \ldots, \zeta^{a_4} x_4)$

where $\zeta = e^{2\pi i/5}$.

G acts on X, and then

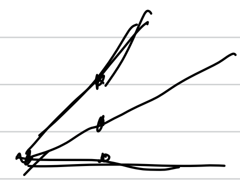
we can take the quotient X/G.

Very singular!

But $\exists$ a resolution of singularities

$\check{X} \longrightarrow X/G$ s.t. $\check{X}$ is also

a C-Y 3-fold.



$\chi_{top}(X) = -200$ $\quad$ $\mathrm{Pic}\, X = \mathbb{Z} = H^2(X, \mathbb{Z})$

$\chi_{top}(\check{X}) = 200$ $\quad$ $\mathrm{Pic}\, \check{X} = \mathbb{Z}^{101} = H^2(\check{X}, \mathbb{Z})$

<u>Candelas, de la Ossa, Green, Parkes '91</u>

Calculations involving <u>period integrals</u>

on $\check{X}$ (Let $\check{\Omega}$ be nowhere vanishing holomorphic 3-form on $\check{X}$.

$$\left(\int_{\alpha \in H_3(\check{X}, \mathbb{Z})} \check{\Omega} \right)$$

gives a generating function for

the numbers of rational curves of each degree on X.

2875 lines on a general quintic X

609,250 conics " " " " "

$\sim 3.17 \times 10^9$ twisted cubics " " " " "

Gromov-Witten invariants

Possible directions:

(1) Try to understand the background and the details necessary for the original physics calculation (covered in reference 3)

(2) Batyrev's construction of mirror pairs as hypersurface in toric varieties. Toric varieties are determined by lattice polytopes, and Batyrev sees mirror symmetry as induced by a duality on polytopes.

(3) Understand something about Gromov-Witten invariants and understand the proof

of Candelas et al's predictions.

Reference #2: Cox + Katz (1999, Mirror Symmetry and Algebraic Geometry.)

(4) Homological mirror symmetry

Kontsevich: mirror symmetry can be explained via an isomorphism between the symplectic geometry of X and the complex geometry of $\check{X}$;

i.e., $\exists$ an isomorphism of categories

See Dirichlet branes + mirror symmetry Chap. 8.

$$\mathrm{Fuk}(X) \cong D(\check{X})$$

Fukaya category of X

Not a category, but an A_∞-category.

Objects are Lagrangian submanifolds.

derived category of coherent sheaves on $\check{X}$.

(5) Investigate approaches involving tropical geometry.

(MG - Tropical geometry and mirror symmetry.)

$$\int_{\substack{M \\ \cap \\ (\mathbb{C}^*)^2 \\ x,y}} e^{W(x,y)/\hbar} \frac{dx \wedge dy}{x \cdot y}$$

Toric varieties: Fulton - Introduction to toric varieties

Toric varieties by

Cox, Little, Schenck