

1. Using geodesic polar coordinates, show that given $p \in S$, we can express the Gaussian curvature as

$$K(p) = \lim_{r \rightarrow 0} \frac{3(2\pi r - L)}{\pi r^3},$$

where L is the length of the geodesic circle of radius r . *Hint: Taylor expansion.*

2. Find the geodesic curvature of a parallel of latitude on the 2-sphere.

3. Prove that on a surface of constant Gaussian curvature, the geodesic circles have constant geodesic curvature. Moreover, prove this in two ways: by direct computation, and by showing that rotation in a geodesic circle is a local isometry (cf. Problem 15 from the last example sheet).

4. Let S be a connected surface and $f, g : S \rightarrow S$ two isometries. Assume that $f(p) = g(p)$ and $df_p = dg_p$. Show that $f(q) = g(q)$ for all $q \in S$.

5. Let p be a point on a surface S . Complete the outline in lecture to show that there exists an open set V containing p such that if $\gamma : [0, 1] \rightarrow V$ is a geodesic with $\gamma(0) = p$ and $\gamma(1) = q$ and $\alpha : [0, 1] \rightarrow S$ is a regular curve joining p to q , then

$$\ell(\gamma) \leq \ell(\alpha)$$

with equality iff α is a reparametrization of γ .

6. Let S be a compact orientable surface which is not diffeomorphic to a sphere. Prove that there are points on S where the Gaussian curvature is positive, negative and zero.

7. Let S be a compact oriented surface with positive Gaussian curvature and let $N : S \rightarrow \mathbb{S}^2$ be the Gauss map. Let γ be a simple closed geodesic in S , and let A and B be the regions which have γ as a common boundary. Show that $N(A)$ and $N(B)$ have the same area.

8. Let S be an orientable surface with Gaussian curvature $K \leq 0$. Show that 2 geodesics γ_1 and γ_2 which start from a point $p \in S$ will not meet again at a point q in such a way that the images of γ_1 and γ_2 form the boundary of a domain homeomorphic to a disk.

9. Let S be a surface homeomorphic to a cylinder with negative Gaussian curvature. Show that S has at most one simple closed geodesic. Does the result remain true if “negative” is replaced by “nonpositive”?

10. Let $\phi : U \rightarrow S$ be an orthogonal parametrization around a point p . Let $\alpha : [0, \ell] \rightarrow \phi(U)$ be a simple closed curve parametrized by arc-length enclosing a domain R . Fix a unit vector $w_0 \in T_{\alpha(0)}(S)$ and consider $W(t)$ the parallel transport of w_0 along α . Let $\psi(t)$ be a differentiable determination of the angle from ϕ_u to $W(t)$. Show that

$$\psi(\ell) - \psi(0) = \int_R K dA.$$

11. Let S_t be a family of smooth oriented surfaces, where t ranges in an interval $I \subset \mathbb{R}$ containing 0, such that, around each point $p \in S_t$, there exists a family of local parametrizations $x_t(u, v), y_t(u, v), z_t(u, v)$ of S_t such that $x(t, u, v)$, etc. are smooth maps and

$$(\dot{x}_t, \dot{y}_t, \dot{z}_t) = H_t(x_t, y_t, z_t)N_t(x_t, y_t, z_t),$$

where \cdot denotes differentiation with respect to t , H_t denotes the mean curvature of S_t , N_t denotes the normal of S_t , and it is assumed that $H_t \neq 0$ and N_t is continuous in t . We say that S_t evolves under mean curvature flow.

Show that the map $\phi_t : S_0 \rightarrow S_t$ defined by taking $(x_0, y_0, z_0) \mapsto (x_t, y_t, z_t)$ is well-defined, i.e. it does not depend on the parametrizations. Show that the map $\phi : I \times S_0 \rightarrow S_t$ is smooth.

Now let γ_0 be a closed geodesic in S_0 , and define $\gamma_t = \phi_t \circ \gamma$. Let $L(\gamma_t)$ denote the length. Show that

$$\frac{d}{dt}L(\gamma_t)|_{t=0} \leq -\frac{4\pi^2}{L(\gamma_0)}.$$

The next 3 problems are non-examinable.

11. Let S be connected. Recall from the lectures the definition of the holonomy group $\text{Hol}(S)$ and the reduced holonomy group $\text{Hol}_0(S)$. (We define $\text{Hol}(S)$ as follows. Pick a point p and consider the set of all maps $T_p(S) \rightarrow T_p(S)$ given by $w_0 \rightarrow W_\alpha(\ell)$, where α ranges over piecewise smooth loops $\alpha : [0, \ell] \rightarrow S$ with $\alpha(0) = \alpha(\ell) = p$, and $W_\alpha(t)$ denotes parallel transport of w_0 along α . This is a subgroup of $O(2)$ and, in the orientable case, a subgroup of $SO(2)$. The group does not depend on the point p . $\text{Hol}_0(S)$ is defined similarly, but with the restriction that α be contractible.)

(i) Show that either $\text{Hol}_0(S) = I$ or $\text{Hol}_0(S) = SO(2)$. Characterize completely the former case.

(ii) Give an example of a connected oriented surface S for which $\text{Hol}(S) \neq I$, $\text{Hol}(S) \neq SO(2)$.

(iii) Give an example of a higher dimensional Riemannian manifold (\mathcal{M}, g) such that $\text{Hol}_0(M) \neq I$, $\text{Hol}_0 \neq SO(n)$.

12. (The Poincaré–Hopf theorem) Let S be an oriented surface and $V : S \rightarrow \mathbb{R}^3$ a smooth vector field, that is, $V(p) \in T_p S$ for all $p \in S$. We say that p is *singular* if $V(p) = 0$. A singular point p is *isolated* if there exists a neighborhood of p in which V has no other zeros. The singular point p is *non-degenerate* if $dV_p : T_p S \rightarrow T_p S$ is a linear isomorphism. (Can you see why dV_p takes values in $T_p S$?) Show that if a singular point is non-degenerate, then it is isolated.

To each singular point p we associate an integer called the *index* of the vector field at p as follows: Let $\phi : U \rightarrow S$ be an orthogonal parametrization around p compatible with the orientation. Let $\alpha : [0, l] \rightarrow \phi(U)$ be a regular piecewise smooth simple closed curve so that p is the only zero of V in the domain enclosed by α . Let $\phi(t)$ be some differentiable determination of the angle from ϕ_u to $V(t) \doteq V \circ \alpha(t)$. Since α is closed, there is an integer I (the index) defined by

$$2\pi I \doteq \phi(l) - \phi(0).$$

(i) Show that I is independent of the choice of parametrization (Hint: use Problem 10). One can also show that I is independent of the choice of curve α , but this is somewhat harder. In addition, one can prove that if p is non-degenerate, then $I = 1$ if dV_p preserves orientation and $I = -1$ if dV_p reverses orientation.

(ii) Give examples of various vector fields on \mathbb{R}^2 with isolated singularities at the origin, computing their indices. Draw pictures.

(iii) Suppose now that S is compact and V is a smooth vector field with isolated singularities. Consider a triangulation of S such that

- every triangle is contained in the image of some orthogonal parametrization,

- every triangle contains at most one singular point
- the boundaries of the triangles contain no singular points and are positively oriented.

Show that

$$\sum_i I_i = \frac{1}{2\pi} \int_S K dA = \chi(S),$$

where I_i denote the indices of the singular points. Thus, you have proved that the sum of the indices of a smooth vector field with isolated singularities on a compact surface is equal to the Euler characteristic. This is known as the *Poincaré–Hopf theorem*. Conclude that a surface homeomorphic to \mathbb{S}^2 cannot be “combed”.

Finally, suppose $f : S \rightarrow \mathbb{R}$ is a Morse function and consider the vector field given by the gradient of f , i.e. $\nabla f(p)$, defined in turn by the relation $\langle \nabla f(p), v \rangle = df_p(v)$ for all $v \in T_p S$. (Show that $\nabla f(p)$ is indeed well defined and only depends on the first fundamental form of S .) Use the Poincaré–Hopf theorem to show that $\chi(S)$ is the number of local maximum and minima minus the number of saddle points. Use this to find the Euler characteristic of a surface of genus two.

13. (The degree of the Gauss map) Let S be a compact oriented surface and let $N : S \rightarrow \mathbb{S}^2$ be its Gauss map. Let $y \in \mathbb{S}^2$ be a regular value of N . Rather than counting the preimages of y modulo 2 as we did in the first lectures, we will count them with sign. Let $N^{-1}(y) = \{p_1, \dots, p_n\}$. Let $\varepsilon(p_i)$ be $+1$ if dN_{p_i} preserves orientation ($K(p_i) > 0$), and -1 if dN_{p_i} reverses orientation ($K(p_i) < 0$). Now let

$$\deg(N) \doteq \sum_i \varepsilon(p_i).$$

As in the case of the degree mod 2, it can be shown that the sum on the right hand side is independent of the regular value and $\deg(N)$ turns out to be an invariant of the homotopy class of N .

Now choose $y \in \mathbb{S}^2$ such that both y and $-y$ are regular values of N . (Why can we do this?) Let V be the vector field on S given by

$$V(p) \doteq \langle y, N(p) \rangle N(p) - y.$$

- Show that the index of V at a zero p_i is $+1$ if dN_{p_i} preserves orientation and -1 if dN_{p_i} reverses orientation.
- Show that the sum of the indices of V equals twice the degree of N .
- Show that $\deg(N) = \chi(S)/2$.

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