

1. Let $\alpha : I \rightarrow S$ be a geodesic. Show that if α is a plane curve and $\ddot{\alpha}(t) \neq 0$ for some $t \in I$, then $\dot{\alpha}$ is an eigenvector of $dN_{\alpha(t)}$ where N denotes the Gauss map. *Hint: compare the normal n of α with the normal N of S .*
2. Show that if all geodesics of a connected surface are plane curves, then the surface is contained in a plane or sphere. *Hint: use the previous problem and the results of the previous example sheet.*
3. Let $f : S_1 \rightarrow S_2$ be an isometry between two surfaces.
 - (i) Let $\alpha : I \rightarrow S_1$ be a regular curve and $V(t)$ a parallel vector field along α . Show that $df_{\alpha(t)}(V(t))$ is a parallel vector field along $f \circ \alpha$. Show more generally that for V not necessarily parallel, $\frac{DV}{dt} = \frac{D(df_{\alpha(t)}V(t))}{dt}$.
 - (ii) Show that f preserves geodesics.
4. Consider the surface of revolution from Problem 10 of the last example sheet.
 - (i) Compute explicitly the differential equation for geodesics.
 - (ii) Establish *Clairaut's relation*: $f^2\dot{u}$ is constant along geodesics. Show that if θ is the angle that a geodesic makes with a parallel and r is the radius of the parallel at the intersection point, then Clairaut's relation says that $r \cos \theta$ is constant along geodesics.
 - (iii) Show that meridians are geodesics; when is a parallel a geodesic?
5. Let S_1 and S_2 be surfaces with Gauss curvature K_{S_1} , K_{S_2} , respectively. Suppose $f : S_1 \rightarrow S_2$ is a diffeomorphism with $K_{S_2}(f(x)) = K_{S_1}(x)$. Is f an isometry?

Now suppose that f maps geodesics of S_1 to geodesics of S_2 . Is f an isometry?
6. Show that there are no compact minimal surfaces in \mathbb{R}^3 .
7. Let S be a connected surface and let p be such that all geodesics through p are closed, i.e. all geodesics through p extend to smooth maps $\gamma : \mathbb{S}^1 \rightarrow S$. Show that S is compact.
8. Let S be a regular surface without umbilical points. Prove that S is a minimal surface iff the Gauss map $N : S \rightarrow S^2$ satisfies

$$\langle dN_p(v_1), dN_p(v_2) \rangle = \lambda(p) \langle v_1, v_2 \rangle$$

for all $p \in S$ and all $v_1, v_2 \in T_p S$, where $\lambda(p) \neq 0$ is a number depending only on p . By considering stereographic projection deduce that isothermal coordinates exist around a nonplanar point in a minimal surface.

9. Let $D \subset \mathbb{C}$, f and g complex functions and $\phi : D \rightarrow \mathbb{R}^3$ the parametrization defined by the Weierstrass representation. Show that ϕ is an immersion iff f vanishes only at the poles of g and the order of its zero at each pole is exactly twice the order of the pole of g .
10. Find D , f , g representing the catenoid $\phi(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av)$ and helicoid $\phi(u, v) = (a \sinh v \cos u, a \sinh v \sin u, av)$.
11. Show that the Gaussian curvature of the minimal surface determined by a Weierstrass representation defined by functions f and g is

$$K = - \left(\frac{4|g'|}{|f|(1+|g|^2)^2} \right)^2.$$

Show that either $K = 0$ identically, or the zeros of K are isolated. [*There is a way of doing this problem almost entirely without calculation. Think about the relation between g and the Gauss map, together with the fact that stereographic projection is conformal.*]

12. The Weierstrass representation is not unique: if $\phi_{(f,g)} : D \rightarrow \mathbb{R}^3$ is the associated parametrization and $\alpha : W \rightarrow D$ is a bijective holomorphic map, then $\phi_{(f,g)} \circ \alpha$ is another representation of the same minimal surface and thus can be expressed as a Weierstrass parametrization corresponding to different f and g . Show that by choosing $\alpha(z) = g^{-1}(z)$, by restricting to a suitable open set around a point which is not a pole of g and for which g' is not zero, we may assume that (f, g) is of the form (F, id) . We denote such a parametrization by ϕ_F .

13. Show that the minimal surfaces given by $\phi_{e^{-i\theta}F}$ for θ real are all locally isometric. With an appropriate choice of F , show that the catenoid and the helicoid are locally isometric.

14. Show that any geodesic of the paraboloid of revolution $z = x^2 + y^2$ which is not a meridian intersects itself an infinite number of times. *Hint: use Clairaut's relation. You may assume that no geodesic of a surface of revolution can be asymptotic to a parallel which is not itself a geodesic.*

15. Let S_1 and S_2 be surfaces and $p_1 \in S_1, p_2 \in S_2$.

(i) Suppose S_1 and S_2 are locally isometric around p_1 and p_2 . Show that there exists a geodesic $\gamma_1(t)$ emanating from p_1 and $\gamma_2(t)$ emanating from p_2 so that $K_{S_1}(t) = K_{S_2}(t)$, and such that, for all θ , if $\alpha_1(t), \alpha_2(t)$ are geodesics emanating from p_1, p_2 respectively, making an angle θ from γ_1, γ_2 , respectively, then $K_{S_1}(t) = K_{S_2}(t)$ along $\alpha_1(t)$ and $\alpha_2(t)$. Here K_{S_i} denotes the Gauss curvature of S_i and geodesics are considered parametrized by arc length.

(ii) Show conversely that if the above property is satisfied then p_1 and p_2 are locally isometric. Deduce that if S_1 and S_2 have constant curvature $K_{S_1} = K_{S_2}$, then S_1 and S_2 are locally isometric. *Hint: consider geodesic polar coordinates and solve an ode for G .*

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