1. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve parametrized by arc-length, with curvature $k(s) \neq 0$ for all $s \in I$. Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\langle \dot\alpha \wedge \ddot\alpha, \dddot\alpha \rangle}{|k(s)|^2}.$$

2. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve parametrized by arc length with $\tau(s) \neq 0$ and $\dot k(s) \neq 0$ for all $s \in I$. Show that a necessary and sufficient condition for $\alpha(I)$ to lie on a sphere is that

$$R^2 + (\dot R)^2 T^2$$

is constant, where $R = 1/k$ and $T = 1/\tau$. *Hint: To prove that the condition is necessary you need to differentiate three times* $|\alpha(s)|^2$. *To prove sufficiency, differentiate* $\alpha + Rn - \dot R T b$.

3. Consider a closed plane curve inside a disk of radius r. Prove that there exists a point on the curve at which the curvature has absolute value $\geq 1/r$.

4. Let $\overline{AB}$ be a straight line segment in the plane with endpoints A and B and let ℓ be a fixed number strictly bigger than the length of $\overline{AB}$. Show that the curve joining A and B with length ℓ and such that together with $\overline{AB}$ bounds the largest possible area is an arc of a circle passing through A and B. You may suppose that the isoperimetric inequality holds for piecewise smooth boundaries.

5. (i) Show that of all n-sided (simple) polygons of equal perimeter, the n-sided regular polygon encloses the most area.

(ii) Show that if $n > m$, then the n-sided regular polygon encloses more area than the m-sided regular polygon of equal perimeter.

Deduce a version of the isoperimetric inequality for a suitably general class of curves from the above statements.

6. Let $\phi : U \to S$ be a parametrization of a surface S in $\mathbb{R}^3$. Show that

$$|\phi_u \wedge \phi_v| = \sqrt{EG - F^2}.$$

7. Let $\alpha : [0, \ell] \to \mathbb{R}^3$ be a curve parametrized by arc length with everywhere non-zero curvature. Suppose α has no self-intersections, $\alpha(0) = \alpha(\ell)$ and it induces a smooth map from $\mathbb{S}^1$ to $\mathbb{R}^3$ (i.e α is a smooth simple closed curve). Let $r > 0$ and consider the map $\phi : [0, \ell] \times [0, 2\pi] \to \mathbb{R}^3$ given by:

$$\phi(s, v) = \alpha(s) + r(n(s)\cos v + b(s)\sin v)$$

where $n = n(s)$ and $b = b(s)$ are the normal and binormal vectors of α. The image T of ϕ is called the *tube* of radius r around α. It can be shown that for r sufficiently small T is a surface. Prove that the area of T is $2\pi r\ell$.

8. (i) Let S be a surface that can be covered by connected coordinate neighborhoods V_1 and V_2. Assume that $V_1 \cap V_2$ has two connected components W_1 and W_2, and that the Jacobian of the change of coordinates is positive on W_1 and negative on W_2. Prove that S is not orientable.

(ii) Let $\phi : [0, 2\pi] \times (-1, 1) \to \mathbb{R}^3$ given by:

$$\phi(u, v) = ((s - v\sin(u/2))\sin u, (s - v\sin(u/2))\cos u, v\cos(u/2)).$$

The image of ϕ is the Möbius strip. By considering the parametrizations given by ϕ restricted to $(0, 2\pi) \times (-1, 1)$ and

$$\psi(\bar{u}, \bar{v}) = ((2 - \bar{v}\sin(\pi/4 + \bar{u}/2))\cos\bar{u}, -(2 - \bar{v}\sin(\pi/4 + \bar{u}/2))\sin\bar{u}, \bar{v}\cos(\pi/4 + \bar{u}/2)),$$

$(\bar{u}, \bar{v}) \in (0, 2\pi) \times (-1, 1)$, show that the Möbius strip is not orientable.

9. Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi}\int_0^\pi k_n(\theta)d\theta,$$

where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

10. Consider a surface of revolution parametrized by $\phi : (0, 2\pi) \times (a, b) \to \mathbb{R}^3$, where

$$\phi(u, v) = (f(v)\cos u, f(v)\sin u, g(v)).$$

Suppose f never vanishes and that $(f(v), g(v))$ is parametrized by arc length. Compute the Gaussian curvature K and the mean curvature H.

11. (i) Determine an equation for the *tractrix*, which is the curve such that the length of the segment of the tangent line between the point of tangency and some fixed line L in the plane–not meeting the curve–is a constant equal to 1.

(ii) Rotate the tractrix about the line L to obtain a surface of revolution (called the *pseudosphere*). Compute its Gaussian curvature.

12. Let S be a compact orientable surface in $\mathbb{R}^3$. Show that the Gauss map is surjective and that it hits almost every direction the same number of times modulo 2. (You may use the Jordan-Brouwer separation theorem.) Show that S always has an elliptic point.

13. If ϕ is an orthogonal parametrization, i.e. $F = 0$, show that the Gauss formula yields:

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right).$$

14. Let p be a point of a surface S such that the Gaussian curvature $K(p) \neq 0$ and let V be a small connected neighborhood of p where K does not change sign. Define the *spherical area* $A_N(B)$ of a domain B contained in V as the area of $N(B)$ if $K(p) > 0$ or as minus the area of $N(B)$ if $K(p) < 0$, where N here denotes the Gauss map. Show that

$$K(p) = \lim_{A \to 0} \frac{A_N(B)}{A(B)}$$

where $A(B)$ is the area of B and the limit is taken through a sequence of domains B_n that converge to p in the sense that any sphere around p contains all B_n for all n sufficiently large. (Note: This is the way Gauss introduced K.)

15. Show that if S is a connected surface in $\mathbb{R}^3$ such that every point is umbilic, then S is part of a plane or a sphere. *Hint: use that in a parametrization* $\phi(u, v)$, $N_{uv} = N_{vu}$.

16. Let S be a connected surface in $\mathbb{R}^3$. Suppose that for each $p \in S$, there exists a Euclidean rotation ρ of $\mathbb{R}^3$ centered at p, such that $\rho^n \neq 1$ for $1 \leq n \leq 5$, and such that $\rho(S) = S$. Show that S is a plane or sphere.

17. Given an example of two curves $\alpha : [0,1] \to \mathbb{R}^3$, $\tilde{\alpha} : [0,1] \to \mathbb{R}^3$, parametrized by arc length, such that $k(s) = \tilde{k}(s)$, and $\tau(s) = \tilde{\tau}(s)$ whenever $k(s) \neq 0$, but such that α and $\tilde{\alpha}$ are not related via a Euclidean motion.

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