1. Let (X, d_X), (Y, d_Y) be metric spaces. Define d on $X \times Y$ by $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$. Show that d makes $X \times Y$ into a metric space. If X and Y are complete, show that $X \times Y$ is complete.

2. Let X, Y be metric spaces. Show that if $\phi : X \to Y$ is continuous, then the graph Γ of ϕ, defined by $\Gamma = \{(x, \phi(x))\}$, is closed.

3. Give an example of metric spaces X and Y, and a map $\phi : X \to Y$, such that ϕ is not continuous, yet the graph of ϕ is closed and bounded.

4. Let X be a metric space. Show that the following are equivalent: X is compact, X is countably compact, X is sequentially compact. (X is countably compact if every countable open cover has a finite subcover.)

5. Let X be a normed vector space, and Y a *closed* proper subspace. Show that for all $\epsilon > 0$, there exists an $x \in X$, such that $||x|| = 1$, and

$$\inf_{y \in Y} ||x - y|| \geq 1 - \epsilon$$

6. Use the above two exercises to give another proof that if X is a normed vector space and $\{x : ||x|| \leq 1\}$ is compact, then X is finite dimensional.

7. Use the Hahn-Banach theorem to show that $(\ell_\infty)^* \neq \ell_1$.

8. Use a category argument to show problem 15 from Example Sheet 1.

9. Let X be a topological space. Show that the space $C(X)$ of complex-valued bounded continuous functions is a Banach space, with the sup norm $||f|| = \sup_{x \in X} |f(x)|$.

10. Prove the following complex version of the Tietze-Urysohn extension theorem: Let X be a normal topological space, and let $A \subset X$ be closed. Suppose $f : A \to \mathbb{C}$ is continuous with $||f|| < \infty$. Then there exists a continuous function $\tilde{f} : X \to \mathbb{C}$ such that $\tilde{f}|_A = f$, and $||\tilde{f}|| = ||f||$.

11. Show that there exist continuous nowhere differentiable functions $f : [0, 1] \to \mathbb{R}$. *Hint:* Exhibit the set of nowhere differentiable functions as a countable intersection of dense open sets in $C[0, 1]$.

12. Let T denote the unit circle in the usual 2-plane, and let $C(T)$ denote the space of complex-valued continuous functions on T. Since T is compact, by Exercise 9, $C(T)$ is a Banach space. We will parametrize T by the angular variable $t \in [-\pi, \pi)$.[1] Define,

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt}$$

and define the operator

$$S_n : C(T) \to C(T) \tag{1}$$

[1] i.e., one can think of $C(T)$ as the space of continuous periodic functions on $\mathbb{R}$ with period 2π

by

$$(S_n(f))(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt.$$

The function $S_n(f)$ is called the n'th partial sum of the Fourier series of f, i.e.

$$S_n(f) = \sum_{k=-n}^{n} \hat{f}(k)e^{ikx},$$

where for $k \in \mathbf{Z}$,

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt. \tag{2}$$

In the next few exercises, we shall examine the question of whether this series necessarily converges to f.

In this exercise, show that S_n is indeed a linear operator with target as given by (1). Show that each $||S_n|| < \infty$. Let ϕ_n denote the composition of S_n with the evaluation-at-0 map $e_0 : C(T) \to \mathbb{C}$ given by $e_0(f) = f(0)$. Show that $||\phi_n|| < \infty$. Thus $\phi_n \in C(T)^*$.

13. Show that in fact,

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{\sin(t/2)}.$$

14. Use this to show that $\int_{-\pi}^{\pi} |D_n| dt \to \infty$ as $n \to \infty$. Deduce that $||\phi_n|| \to \infty$. Using the Banach-Steinhaus Theorem, deduce that there exists a function $f \in C(T)$ whose Fourier series diverges at 0.

15. Show that in fact, there exists a dense set $\mathcal{F} \subset C(T)$, such that for each $f \in \mathcal{F}$, there exists a dense set $S_f \subset (-\pi, \pi)$ such that the Fourier series of f diverges everywhere on S_f.

16. Let $f \in C(T)$. Show that $\hat{f}(n) \to 0$, where $\hat{f}$ is defined by (2). Thus, $f \to \hat{f}$ defines a linear map $\Lambda : C(T) \to c_0$, where c_0 is defined[2] as the Banach space of functions $\mathbb{Z} \to \mathbb{C}$, with norm given by

$$||\hat{g}||_\infty = \sup_{n\in\mathbb{Z}} |\hat{g}(n)|.$$

17. For those who know about Lebesgue measure: Define the space $L^1(T)$ as the set of all Lebesgue integrable functions on T, modulo null functions. Define

$$||f||_1 = \int_{-\pi}^{\pi} |f(t)|dt.$$

This norm makes L^1 into a Banach space. Show that Λ defined by $f \to \hat{f}$, where again $\hat{f}$ is defined by (2), taken now in the sense of Lebesgue, maps

$$\Lambda : L^1 \to c_0.$$

Show moreover that this map is bounded and injective.

18. Show that $D_n(t)$ defined previously satisfies $||D_n||_1 \to \infty$, while on the other hand $||\Lambda(D_n)||_\infty = 1$. Deduce from the inverse mapping theorem that Λ is *not surjective*.

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[2]Compare with the definition in Example Sheet 1.