1. Let $X = \mathbb{C}^2$, and let $T(x)$ be the linear map taking (x_1, x_2) to $(x_1 + x_2, x_1)$. Compute the spectrum $\sigma(T)$, the point spectrum $\sigma_p(T)$, and the resolvent set $\rho(T)$.

2. Let H be a Hilbert space, let $T : H \to H$ a bounded linear map, and let $T^* : H \to H$ denote its adjoint. Show that $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$.

3. Reinterpret problem 15 of the previous example sheet as the statement that a certain operator on a Hilbert space is compact.

4. Let $T \in \mathcal{B}(X)$. Define the *spectral radius* $r(T)$ by

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|.$$

Show that if S, T commute, then $r(S+T) \leq r(S) + r(T)$, and $r(ST) \leq r(S)r(T)$. Construct a Banach space X and noncommuting operators S, T violating these inequalities.

5. Let $f \in C(\mathbf{T})$, where $\mathbf{T}$ denotes the unit circle in the complex plane, parametrized by $t \in [0, 2\pi)$ via the map e^{it}. Show that

$$\lim_{|n| \to \infty} \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt = 0.$$

6. Let X be a Banach space, and $T : X \to X$ a bounded linear operator. Define the *compression spectrum* $\sigma_{com}(T) \subset \sigma(T)$ to be the set $\lambda \in \sigma(T)$ such that $\overline{Im(\lambda I - T)} \neq X$. Define the *residual spectrum* to be the set $\sigma_r(T) = \sigma_{com} \setminus \sigma_p$. Finally define the *continuous spectrum* $\sigma_c(T) = \sigma(T) \setminus (\sigma_r(T) \cup \sigma_p(T)$. Show that any $\lambda \in \sigma_c(T)$ is necessarily an *approximate eigenvalue*, i.e., there exists a sequence of vectors $x_i \in X$, $||x_i|| = 1$, such that $Tx_i - \lambda x_i \to 0$.

7. Let $\emptyset \neq K \subset \mathbf{C}$ be compact. Construct a Hilbert space H and an operator $T : H \to H$ such that $K = \sigma(T)$.

8. Let H be a Hilbert space, and $T : H \to H$ a bounded linear operator. Suppose that T^*T is compact, where T^* denotes the adjoint. Show that T is compact.

9. Let X be a Banach space, and T a bounded linear operator $T : X \to X$. Define the *numerical range* of T to be the subset of $\mathbf{C}$ defined by:

$$V(T) = \{f(Tx) : x \in X, f \in X^*, ||x|| = ||f|| = f(x) = 1\}.$$

Show that

$$\sigma(T) \subset \overline{V(T)}$$

Show that if $X = H$ a Hilbert space, and T is hermitian, then

$$\overline{V(T)} = co(\sigma(T)),$$

where $co(A)$ denotes the *convex hull* of A, i.e., the set of all convex combinations $ta_1 + (1-t)a_2$ as a_i range over A.

10. Construct a bounded linear operator T on the space ℓ_p, $p \geq 1$, such that $\sigma(T) = \{0\}$, $\sigma_p(T) = \emptyset$.

11. Give an interesting example of a compact operator on a Hilbert space, not trivially related to the examples considered in this example sheet or in class.

12. Let $T : H \to H$ be a compact Hermitian operator. Prove the *Fredholm alternative*: Let $\lambda \in \mathbf{R}$, $\lambda \neq 0$, and let $x_0 \in H$. Consider the equations

$$Tx = \lambda x \tag{1}$$

$$Tx = \lambda x + x_0 \tag{2}$$

Then, either (1) has no non-zero solutions and (2) has a unique solution x, or else (2) has a solution iff $x_0 \perp N_\lambda$, where N_λ denotes the space of solutions of (1), and the space of solutions of (2) has dimension that of N_λ.

13. Let X be a Banach space, and let $U \subset \mathbf{C}$ be open, and let

$$f : U \to X.$$

We will say f is *analytic* if for all $z_0 \in U$, there exists an open neighborhood $V \subset U$ of z_0 such that $f|_Z$ can be written

$$f = \sum_{n=0}^{\infty} A_n (z - z_0)^n$$

where $A_n \in X$, and the series converges absolutely, i.e.

$$\sum ||A_n (z - z_0)^n||$$

converges, where $|| \cdot ||$ denotes the norm of X. Prove *Liouville's theorem*: If f is bounded and analytic, and $U = \mathbb{C}$, then f is constant.

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