1. Show that in a Hausdorff space, all finite sets are closed. Give an example of a normal space which is not Hausdorff.

2. Let $\epsilon > 0$. Show that the Taylor series of the function $\sqrt{t+\epsilon^2}$ about $t = \frac{1}{2}$ converges uniformly on $[0, 1]$. Recall how this was used in the proof of the Stone-Weierstrass theorem.

3. Prove the complex version of Stone-Weierstrass: Let L be a locally compact Hausdorff space, and let $\mathcal{A} \subset C_0(L)$ be a subalgebra which is closed under complex conjugation and strongly separates points. (Recall that $\mathcal{A}$ is said to *strongly separate points* if $\forall x, y \in L, \exists f \in \mathcal{A} : 0 \neq f(x) \neq f(y) \neq 0$.) Then $\mathcal{A}$ is dense in $C_0(L)$, i.e. $\overline{\mathcal{A}} = C_0(L)$.

4. Prove using Arzela-Ascoli that the initial value problem

$$x' = f(x,t),$$

$$x(0) = x_0,$$

has a local solution $x : (-\epsilon, \epsilon) \to \mathbb{R}$ for sufficiently small ϵ, where $f(\cdot, \cdot)$ is a continuous real-valued function of its arguments, and $x_0 \in \mathbb{R}$ is a constant. Show by example that solutions are not necessarily unique.

5. Let V be a normed space. Show that V is Euclidean if and only if

$$||u+v||^2 + ||u-v||^2 = 2||u||^2 + 2||v||^2$$

for all $u, v \in V$. The above identity is known as the *parallelogram law.*

6. Recall the Banach spaces ℓ_p, for $\infty \geq p \geq 1$. Show that ℓ_p is a Hilbert space if and only if $p = 2$.

7. Let E be a Euclidean space, and let P be a linear map $P : E \to E$ with $0 \neq ||P|| < \infty$, and such that $P^2 = P$. Show that $||P|| = 1$ if and only if P is an orthogonal projection, i.e. if and only if $ImP \perp KerP$.

8. Construct a Euclidean space E and a closed subspace F such that $F + F^{\perp} \neq E$.

9. Let H be a Hilbert space, and $\{e_i\}$ an orthonormal basis. Show that $\{e_i\}$ is closed and bounded. Show that $\{e_i\}$ is compact if and only if H is finite dimensional.

10. Let H be a Hilbert space, let F be a closed subspace of H, and let f be a bounded linear functional on F. Show–without applying the Hahn-Banach theorem–that H can be extended to a bounded linear functional on H, with the same norm.

11. Let $\mathbf{X} \subset \mathbb{R}^n$ be compact. Let $C^n(\mathbf{X})$ denote the space of all functions $f : \mathbf{X} \to \mathbb{R}$ with continuous derivatives up to order n. (By convention, $C^0(\mathbb{R}^n)$ will be the space of all continuous functions.) Define a norm on C^n by

$$||f||_n = \sum_{i=0}^{n} \sup_{x \in \mathbf{X}} |f^{(i)}(x)|$$

Show that this indeed defines a norm, and that C^n is a Banach space. There exists a natural map $\phi : C^n \to C^{n-1}$ which takes a function f to itself, in view of the fact that a C^n function is *a fortiori* C^{n-1}. Let B_n denote the closed unit ball of C^n. Show that $\overline{\phi(B_n)}$ is compact in C^{n-1}. Describe the set $\overline{\phi(B_n)}$.

12. Define a Euclidean space as follows: Let the underlying set be $C[0,1]$, i.e. the continuous complex-valued functions on the interval $[0,1]$, and define an inner product by

$$(f,g) = \int_0^1 f(t)\overline{g(t)}dt.$$

Show that this is not a Hilbert space.

13. Let H be a Hilbert space, and $\emptyset \neq C \subset H$ be a closed convex subset. Let $x_0 \in H$ be fixed. Show that there exists an $x \in C$ such that

$$d(x, x_0) = \inf_{y \in C} d(y, x_0).$$

Show that x is unique, i.e.

$$d(x, x_0) < d(y, x_0)$$

for all $y \in C$, $y \neq x$. Need this be true for general Banach spaces?

14. Let H_1 and H_2 be two Hilbert spaces. Show that one of them is isomorphic to a subspace of the other.

15. Let x_n be an orthonormal system in H. Let $a_n \in \ell_2(\mathbb{R})$, with $a_n \geq 0$. Define $K \subset H$ to be the set of vectors that can be written

$$x = \sum_{n=1}^{\infty} b_n x_n$$

where $b_n \in \mathbb{C}$, $|b_n| \leq a_n$. Show that the series above converges indeed to an element $x \in H$. Then show that K is compact.

For comments, email `M.Dafermos@dpmms.cam.ac.uk`