1. Show that in a Hausdorff space, all finite sets are closed. Give an example of a normal space which is not Hausdorff.

2. Let $\epsilon > 0$. Show that the Taylor series of the function $\sqrt{t+\epsilon^2}$ about $t = \frac{1}{2}$ converges uniformly on $[0,1]$. Don't be afraid to use complex analysis. Recall how this was used in the proof of the Stone-Weierstrass theorem.

3. Consider the complex subalgebra $\mathcal{A} \subset C_{\mathbb{C}}(\mathbb{S}^1)$ generated by $\{e^{in\theta}\}$, where $n \in \mathbb{N}$. Show that $\mathcal{A}$ is a unitial algebra separating points, but $\overline{\mathcal{A}} \neq C_{\mathbb{C}}(\mathbb{S}^1)$.

4. Prove directly using Arzela-Ascoli that the initial value problem

$$x' = f(x,t),$$

$$x(0) = x_0,$$

has a local solution $x : (-\epsilon, \epsilon) \to \mathbb{R}$ for sufficiently small ϵ, where $f(\cdot,\cdot)$ is a continuous real-valued function of its arguments, and $x_0 \in \mathbb{R}$ is a constant. (Do not use the well-posedness theorem in the Lipschitz case.)

5. Let V be a normed space. Show that V is Euclidean if and only if

$$||u+v||^2 + ||u-v||^2 = 2||u||^2 + 2||v||^2$$

for all $u, v \in V$. The above identity is known as the *parallelogram law.*

6. Recall the Banach spaces ℓ_p, for $\infty \geq p \geq 1$. Show that ℓ_p is a Hilbert space if and only if $p = 2$.

7. Let E be a Euclidean space, and let P be a linear map $P : E \to E$ with $0 \neq ||P|| < \infty$, and such that $P^2 = P$. Show that $||P|| = 1$ if and only if P is an orthogonal projection, i.e. if and only if $ImP \perp KerP$.

8. Construct a Euclidean space E and a closed subspace F such that $F + F^{\perp} \neq E$.

9. Let H be a Hilbert space, and $\{e_i\}$ an orthonormal basis. Show that $\{e_i\}$ is closed and bounded. Show that $\{e_i\}$ is compact if and only if H is finite dimensional.

10. Let H be a Hilbert space, let F be a closed subspace of H, and let f be a bounded linear functional on F. Show–without applying the Hahn-Banach theorem–that H can be extended to a bounded linear functional on H, with the same norm.

11. Let $\mathcal{U} \subset \mathbb{R}^n$ be open and bounded and consider $C^n(\overline{\mathcal{U}})$. Let B_n denote the closed unit ball of C^n. Show that $\overline{\phi(B_n)}$ is compact in C^{n-1}. Describe the set $\overline{\phi(B_n)}$.

12. Define a Euclidean space as follows: Let the underlying set be $C[0,1]$, i.e. the continuous complex-valued functions on the interval $[0,1]$, and define an inner product by

$$(f,g) = \int_0^1 f(t)\overline{g(t)}dt.$$

Show that this is not a Hilbert space.

13. Let H be a Hilbert space, and $\emptyset \neq C \subset H$ be a closed convex subset. Let $x_0 \in H$ be fixed. Show that there exists an $x \in C$ such that

$$d(x, x_0) = \inf_{y \in C} d(y, x_0).$$

Show that x is unique, i.e. $d(x, x_0) < d(y, x_0)$ for all $y \in C$, $y \neq x$. Need this be true for general Banach spaces?

14. Let H_1 and H_2 be two Hilbert spaces. Show that one of them is isomorphic to a subspace of the other.

15. Let x_n be an orthonormal system in H. Let $a_n \in \ell_2(\mathbb{R})$, with $a_n \geq 0$. Define $K \subset H$ to be the set of vectors that can be written

$$x = \sum_{n=1}^{\infty} b_n x_n$$

where $b_n \in \mathbb{C}$, $|b_n| \leq a_n$. Show that the series above converges indeed to an element $x \in H$. Then show that K is compact.

16. Let $\mathcal{U} \subset \mathbb{C}$ be open, and let $f_i : \mathcal{U} \to \mathbb{C}$ be a sequence of analytic functions such that $|f_i| \leq M$ for some M. Show that there exists a subsequence f_{i_k} and an analytic $f : \mathcal{U} \to \mathbb{C}$ such that for all $K \subset \mathcal{U}$ compact, $f_{i_k} \to f$ in $C_{\mathbb{C}}(K)$.

17. Now let $B(1) \subset \mathbb{R}^2$ denote the unit open ball, let $\mathcal{F} \subset B_{C^1(\overline{B(1)})}(R) \cap C^n(B(1))$, for some $R > 0$ and an $n \geq 1$, where $B_{C^1(\overline{B(1)})}(R)$ denotes the open R-ball in $C^1(\overline{B(1)})$, and let $\mathcal{G} \subset B_{C^2(\partial B(1))}(R)$, where $\partial B(1)$ is parametrized by θ in the usual way, and differentiability is defined accordingly. All functions are taken here to be real valued. Let $\lambda > 0$, and consider the set $\mathcal{S}_\lambda \subset C^2(\overline{B(1)})$ of all functions ϕ, C^{2+n} in $B(1)$, such that

$$\triangle\phi - \lambda\phi \in \mathcal{F}, \tag{1}$$

$$\phi|_{\partial B(1)} \in \mathcal{G}. \tag{2}$$

Here $\triangle$ denotes the Laplacian $\frac{d^2}{dx^2} + \frac{d^2}{dy^2}$.

Deduce from (1) and (2) that there exists a C, such that $|\nabla\phi| \leq C$ along $\partial B(1)$ for all such ϕ. By considering local maxima and minima for ϕ, $\partial_x\phi$, $\partial_y\phi$ in $B(1)$, show that there exists a constant $\tilde{C}$ such that $|\phi| \leq \tilde{C}$, $|\nabla\phi| \leq \tilde{C}$ in $B(1)$ for all such ϕ. Deduce that $\mathcal{S}_\lambda$ is a totally bounded subset of $C(\overline{B(1)})$. What about the case $\lambda = 0$?

By multiplying $\phi \in \mathcal{S}_\lambda$ by a suitable cutoff function, show that for any point $x_0 \in B(1)$, all derivatives of ϕ up to order n are bounded by some $\hat{C}(x_0, n)$, independent of ϕ, in a neighborhood $B(\epsilon) + x_0$.

18. Let $f \in C^2_{\mathbb{C}}(\partial B(1))$. Show that there exists a function $\phi \in C_{\mathbb{C}}(\overline{B(1)})$, C^∞ in $B(1)$, solving $\triangle\phi = 0$ with $\phi|_{\partial B(1)} = f$, by noting that $\psi(r, \theta) = r^{|m|}e^{im\theta}$ solves $\triangle\psi = 0$ in $B(1)$ for $m \in \mathbb{Z}$, using the fact that the algebra generated by $\{e^{im\theta}\}_{m\in\mathbb{Z}}$ is dense in $C_{\mathbb{C}}(\partial B(1))$, and the compactness which follows from the previous problem in the case $\lambda = 0$. What is the situation if f is assumed merely continuous? Relate to pointwise convergence of Fourier series.

Characterize $\overline{\mathcal{A}}$ of Problem 3 in terms of boundary values of a certain class of functions $\phi : B(1) \to \mathbb{C}$.

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