1. Given an example of a smooth manifold which is not Hausdorff. Give an example which is not paracompact. Give an example which is paracompact but not second countable. From now on, manifolds are assumed Hausdorff and paracompact.

2. Let $F : \mathcal{M} \to \mathcal{N}$ be smooth, let $f \in C^\infty(\mathcal{N})$. Let $F^*(f)$ denote $f \circ F$. Show that $F^*(df) = d(F^*(f))$. Now let $G : \mathcal{N} \to \mathcal{N}'$, and let $x \in \mathcal{M}$. Show that $((G \circ F)_*)_p = (G_*)_{F(p)} \circ (F_*)_p$. Let $\omega \in \Gamma(T^*\mathcal{N}')$ be a 1-form. Show that $(G \circ F)^*\omega = F^*(G^*(\omega))$.

3. Multilinear algebra. Show that the identification of $u^* \otimes v$ with the map sending $u \mapsto u^*(u)v$ extends by linearity an isomorphism $U^* \otimes V \cong Hom(U, V)$. Show that the identification of $u^* \otimes v^*$ with the map sending $u \otimes v \mapsto u^*(u)v^*(v)$ extends to an isomorphism $U^* \times V^* \cong (U \otimes V)^*$. Show that these isomorphisms lead to a natural identification

$$End(U) \cong (U^* \otimes U)^* \cong (End(U))^*$$

where $End(U)$ denotes $Hom(U, U)$. Show that the image of $id \in End(U)$ in $(U^* \otimes U)^*$, under this isomorphism, is the map $C : U^* \otimes U \to \mathbb{R}$ which takes $u^* \otimes u \mapsto u^*(u)$. Show that the image of $id \in End(U)$ in $(End(U))^*$ is the map taking $L \in End(U)$ to trL.

4. Let $\mathcal{M}$ be smooth of dimension m, and let $f_1, \ldots, f_d$ be a collection of smooth functions on $\mathcal{M}$. Let $\mathcal{N}$ denote the set where $f_1 = \cdots = f_d = 0$. Suppose $(df_i)_p$ span a subset of dimension d' in $T^*_p(\mathcal{M})$ for all $p \in \mathcal{N}$. (We assume d' to be constant, but d' need not equal d.) Show that $\mathcal{N}$ can be given the structure of a closed submanifold of $\mathcal{M}$ of dimension $m - d'$.

The above applies to $\mathbb{S}^n$, where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is given by $f(x_1, \ldots, x_{n+1}) = -1 + \sum_{i=1}^{n+1} x_i^2$. Show that the manifold structure defined above is the same as the structure defined via the projection maps to the coordinate hyperplanes.

5. Assuming the classical existence, uniqueness, and continuous dependence on parameters theorem for o.d.e.'s, prove the following version on manifolds: Let $\mathcal{M}$ be a smooth manifold, and let $X \in \Gamma(T\mathcal{M})$ be a smooth vector field. Then for each $x \in \mathcal{M}$, there exists a unique maximal smooth curve $\gamma : (T_-, T_+) \to \mathcal{M}$, with $-\infty \leq T_- < 0 < T_+ \leq \infty$, such that $\gamma(0) = x$ and $\gamma'(t) = X$ for all $t \in (T_-, T_+)$, where $\gamma'(t)$ denotes $(\gamma_*)_t \left(\frac{\partial}{\partial t}\right)$. Moreover, if $T_+ < \infty$, then for every compact $K \subset \mathcal{M}$, there exits a $t_K < T_+$ such that $x[t_K, T_+) \cap K = \emptyset$. To remember the dependence on x, let us denote γ by γ_x, and T_+, T_- by $T_+(x)$, $T_-(x)$. Finally, for every $x \in \mathcal{M}$, there exists an open subset $\mathcal{U}_x$ and an $\epsilon > 0$, such that $(T_-(\tilde{x}), T_+(\tilde{x})) \supset (-\epsilon, \epsilon)$ for all $\tilde{x} \in \mathcal{U}_x$, and such that the map $\phi : \mathcal{U} \times (-\epsilon, \epsilon) \to \mathcal{M}$ defined by $\phi(\tilde{x}, t) \to \gamma_{\tilde{x}}(t)$ is smooth.

6. Let G be a group, and $\mathcal{M}$ a manifold. Let $\mathrm{Diff}(\mathcal{M})$ denote the set of all smooth transformations of $\mathcal{M}$. (Recall that a smooth transformation is a diffeomorphism from $\mathcal{M}$ to itself.) Show that $\mathrm{Diff}(\mathcal{M})$ defines a group with composition as multiplication. Suppose there exists a group homomorphism $R : G \to \mathrm{Diff}(\mathcal{M})$. Given $x \in \mathcal{M}$, we define the *isotropy group* of x as the set of $g \in G$ such that $R(g)x = x$. Show that this defines a subgroup of G, denoted G_x.

We say that R is *properly discontinuous* if the following are true:

1. For all $x, \tilde{x} \in \mathcal{M}$ such that there does not exist a g with $R(g)(x) = \tilde{x}$, there exist neighborhoods $x \in \mathcal{U}$, $\tilde{x} \in \tilde{\mathcal{U}}$, with $R(g)(\mathcal{U}) \cap \tilde{\mathcal{U}} = \emptyset$, for all g.

2. G_x is finite for all $x \in \mathcal{M}$.

3. For all $x \in \mathcal{M}$, there exists a neighborhood $x \in \mathcal{U}$ such that $h(\mathcal{U}) \subset \mathcal{U}$ for all $h \in G_x$, and $\mathcal{U} \cap R(g)(\mathcal{U}) = \emptyset$ for $g \notin G_x$.

Show the following: If $R : G \to \mathrm{Diff}(\mathcal{M})$ is properly discontinuous and injective, then the quotient space $\mathcal{M}/G$ (defined by the equivalence relation $x \sim \tilde{x}$ if there exists a $g \in G$ such that $R(g)x = \tilde{x}$) inherits the structure of a smooth manifold such that the quotient map $\pi : \mathcal{M} \to \mathcal{M}/G$ is smooth.

7. Let $\mathcal{M}$ be a smooth manifold and let x^i denote a local coordinate system on $\mathcal{M}$. Show that (x^i, p^i) define local coordinates on $T\mathcal{M}$, where these are defined by $V_q \mapsto (x^i(q), p^i(V_q))$, where the latter are defined by $V_q = p^i \frac{\partial}{\partial x^i}$. Now let g be a Riemannian metric on $\mathcal{M}$. Show that g induces a Riemannian metric $\tilde{g}$ on the tangent bundle $T\mathcal{M}$ defined in local coordinates as follows:

$$g\left(a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial p^i}, c^i \frac{\partial}{\partial x^i} + d^i \frac{\partial}{\partial p^i}\right) = (a^i c^j + b^i d^j) g_{ij}.$$

8. Take a thin strip of paper and attach the short ends to each other with the opposite orientation. Now exhibit this construction as a rank 1 vector bundle $E \to \mathbb{S}^1$, i.e. a vector bundle whose fibres have dimension 1. We call this the Möbius strip.

We say that two vector bundles E', and E are *equivalent* if there exists a smooth $\phi : E \to E'$ such that $\phi(E_x) = E'_x$ and $\phi|_{E_x}$ is a linear isomorphism of E_x with E'_x. Show that the previous vector bundle is not equivalent to $\mathbb{S}^1 \times \mathbb{R}^1$.

9. Show that a Riemannian metric defines an equivalence, in the above sense, between the bundles $T^*\mathcal{M}$ and $T\mathcal{M}$. What about the converse?

10. Let $F : \mathcal{N}^n \to \mathcal{M}^m$ denote an immersion. Show that there exists a vector bundle E of rank $m - n$ over $\mathcal{N}$, and a smooth map $\tilde{F} : E \oplus T\mathcal{N} \to T\mathcal{M}$, of the form $\tilde{F} : e_p \oplus v_p \mapsto L_p(e_p) + (F_*)_p(v_p)$, where L_p maps linearly $E_p \to T_{F(p)}\mathcal{M}$, and such that $\tilde{F}$ is an isomorphism when restricted to the fibres. Show that this defines E uniquely up to equivalence. We call E the normal bundle of $\mathcal{N}$ defined by F.

Show that in the case of Ex. 4, if $d = d'$, then the normal bundle of $\mathcal{N}$ in $\mathcal{M}$ is trivial.

11. Is $T\mathbb{S}^2$ equivalent to $\mathbb{S}^2 \times \mathbb{R}^2$?