1. Let X be a normed vector space, and let $T : Y \to Z$, $S : X \to Y$ be bounded linear maps. Show that $T \circ S$ is bounded with

$$||T \circ S|| \leq ||T|| \cdot ||S||.$$

Show by specific example that equality need not hold above.

2. Give an example of Banach spaces X and Y, and a bounded linear map $T : X \to Y$ such that $||Tx|| < ||T||$ for all $0 < ||x|| \leq 1$. Can X be finite dimensional?

3. Let $T : X \to Y$, $\tilde{T} : X \to Y$ and $S : Y \to Z$ be bounded linear maps. Show that $(\alpha T + \beta\tilde{T})^* = \alpha T^* + \beta\tilde{T}^*$, where α and β are scalars. Show that $(S \circ T)^* = T^* \circ S^*$.

4. Let $p > 0$, and define the space ℓ_p of all complex sequences such that $\sum_i |x_i|^p < \infty$. Define

$$||x||_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}. \tag{1}$$

For $1 > p > 0$, is this a vector space? Again, for $1 > p > 0$, is this a normed vector space with norm defined by (1)?

5. Define the space ℓ_∞ consisting of all sequences $\{x_i\}$ such that $\sup |x_i| < \infty$, with

$$||\{x_i\}||_\infty = \sup |x_i|.$$

Show that this defines a normed vector space. Recall that a metric space is said to be separable if there exists a countable dense set. Show that ℓ_∞ is not separable.

6. Let $T : V \to W$ be a linear map between finite dimensional normed vector spaces, let e_i denote a basis for V, let $\hat{e}_j$ denote a basis for W, and let a_{ij} denote the components of the matrix representing T in this basis. Determine a basis for W^* and V^* for which T^* has a nice form, and give that form.

7. Let $0 \leq t \leq 1$. Let a and b be nonnegative real numbers. Prove $a^t b^{1-t} \leq ta + (1-t)b$.

8. Let $p \geq 1$, and define q by the relation $p^{-1} + q^{-1} = 1$, with the convention that if $p = 1$, $q = \infty$. We call p and q conjugate exponents. Show that if $p > 1$, and if $x = \{x_i\}$ and $y = \{y_i\}$ are elements of ℓ_p and ℓ_q, respectively, then $xy = \{x_i y_i\}$ is in ℓ_1 and *Hölder's inequality* holds, i.e.

$$||xy||_1 \leq ||x||_p ||y||_q.$$

Show that ℓ_p, for all $p \geq 1$, is a normed vector space with norm defined by (1).

9. Show that for $1 \leq p < \infty$, $\ell_p^* = \ell_q$.

10. Denote by c_0 the subset of ℓ_∞, consisting of all sequences tending to 0. Show that $c_0^* = \ell_1$. Show that ℓ_p is a Banach space for all $1 \leq p \leq \infty$.

11. Let S denote the shift map on ℓ_p, i.e. the map $S : \ell_p \to \ell_p$ taking $(x_1, x_2, \ldots)$ to $(0, x_1, x_2, \ldots)$. Describe S^*. Now for $\infty > p \geq 1$, q conjugate, and $y \in \ell_q$, let T denote the map $T : \ell_p \to \ell_1$ taking $\{x_i\}$ to $\{x_i y_i\}$. Describe T^*.

12. Let X be a vector space, and let $\{p_i\}$ be a countable collection of seminorms, such that for all $0 \neq x \in X$, there exists an i such that $p_i(x) > 0$. (A seminorm is a function $p : X \to \mathbb{R}^+ \cup \{0\}$ that satisfies the axioms of a norm, except positive definitivity, i.e. it is not necessarily the case that $p(v) = 0$ implies $v = 0$.) Fix $1 \leq p \leq \infty$, and define

$$||x|| = ||\{p_i(x)\}||_p,$$

where the right hand side denotes the ℓ_p norm. Let Y denote the subset of X consisting of all x such that the above is finite. Does $|| \cdot ||$ endow Y with the structure of a normed vector space?

13. Let X be a Banach space such that its dual is reflexive. Show that X itself is reflexive.

14. Recall the space ℓ_2^n, i.e. the normed vector space $\mathbb{C}^n$ with norm defined by

$$|(x_1, \ldots, x_n)| = \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

Let $T : \ell_2^n \to \ell_2^n$ be a linear map. Describe $||T||$ algebraically.

15. Let V be a vector space with a countably infinite basis. Show that V cannot be made into a Banach space.

16. A topological vector space V is said to be *locally compact* if there exists a neighborhood of the origin with compact closure. Show that any locally compact topological vector space is finite dimensional.

17. Let $\Omega \subset \mathbb{R}^n$ be *open*, and let $\mathcal{C}(\Omega)$ denote the set of continuous functions on Ω. Make $\mathcal{C}(\Omega)$ into a topological vector space as follows. Choose a sequence of compact K_i such that $\cup K_i = \Omega$, and $K_i \subset K_{i+1}$. Let

$$V(i, n) = \{f : |f|_{\mathcal{C}(K_i)} < 1/n\},$$

and consider the topology on $\mathcal{C}(\Omega)$ generated by this family and all its translates. Show that this defines on $\mathcal{C}(\Omega)$ the structure of a locally convex topological vector space. Show that this topology is not normable. Is it metrizable, i.e. does there exist a metric d on $C(\Omega)$ inducing the above topology?

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