Representations of finite groups of Lie type

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Introduction

It is well known that symmetry is ubiquitous in nature and that its influence pervades all branches of science in one form or another. By abstracting and axiomatizing the salient features of symmetry, the pure mathematician seeks to separate its more tangible side — such as the familiar transformations of reflection and rotation of geometric shapes that preserve the form of the shape — from less 'context specific' aspects, such as the abstract notion that any symmetry transformation has an inverse transformation, and that composing one symmetry transformation with another is again a symmetry transformation. This is made precise in the area of mathematics known as *group theory*, where one considers collections of symmetry transformations in terms of how two of them combine to give a third. In some sense this can be viewed as a generalization of ordinary addition or multiplication of numbers: you add two numbers to get a third. This more abstract point of view can be very powerful indeed since one is able to forget about the complexities of geometry. One may then, if one chooses, apply the information gotten by studying abstract groups back to 'real life' problems involving symmetry. The Frenchman Évariste Galois in the 1830's was the first to do this when he proved theorems about the solutions of polynomial equations by applying group theory to the symmetry exhibited by the solutions of such equations. A closely related field is *representation theory*, where the aim is to take an abstract group and then determine the possible geometric structures which it can operate on. This is not only a useful approach for studying groups themselves, but also has profound influences in many other areas of mathematics and physical sciences such as chemistry and physics.

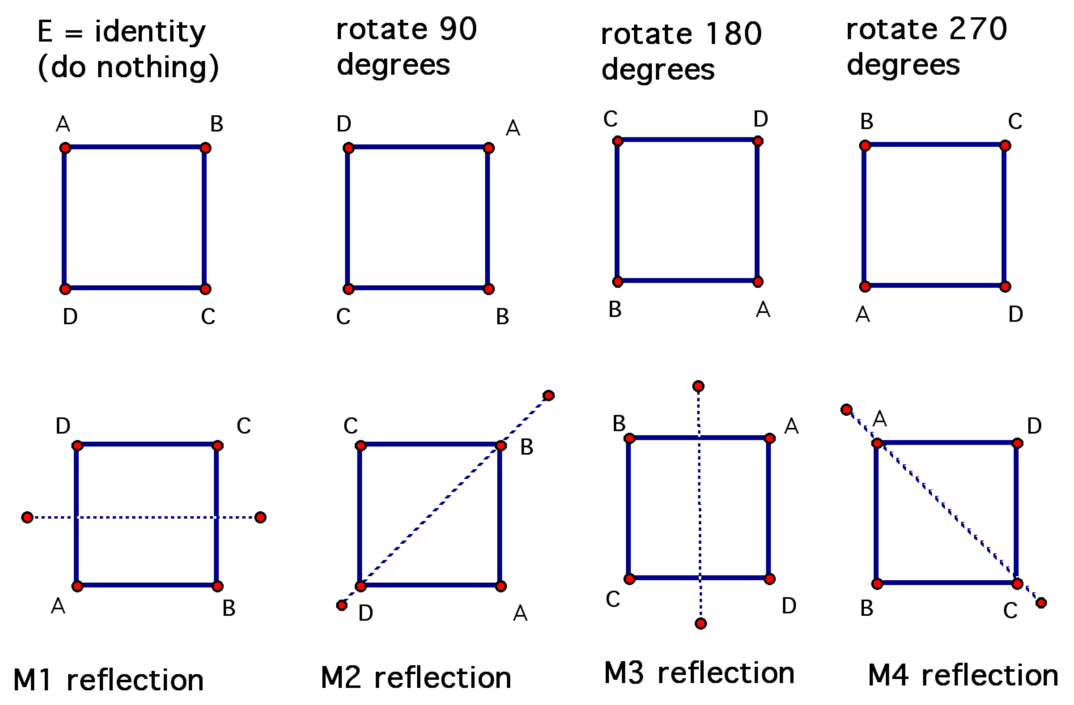

Figure 1: The eight symmetries of a square.

Mathematical definition of a group

Formally, a *group* is a set, usually denoted G, of objects (called the *elements of* G), together with a multiplication rule for combining them. We may denote the elements of G using symbols, say $a, b, c \dots$, and then write multiplication of two elements a, b as $a \cdot b$. In order to be called a group G must satisfy the following axioms:

Closure

For any two elements a, b in G, $a \cdot b$ is also in G.

Associativity

The equation $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ must hold for all elements a, b, c in G.

Identity

There must exist an element, usually denoted by 1, which has the property that $1 \cdot a = a \cdot 1 = a$ for any element a in G.

Inverses

For any element a in G, there must exist an element, usually denoted a^{-1}, which has the property $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

Familiar examples

You will have met many examples of groups already, perhaps without knowing it. It is easy to see, for instance, that the set of integers

$$\dots -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3, \dots$$

is a group if $\cdot$ is taken to be addition. However, if we try using multiplication of numbers as $\cdot$, then we would **not** get a group, since the inverse of 2, say, is $1/2$, which is not an integer, thus the closure axiom fails. On the other hand, if we expanded the integers to the *rational numbers*, i.e. numbers that can be written as fractions, then this problem would go away and indeed the rational numbers with ordinary multiplication is a group.

Some geometry

In the above examples, the set of group elements was infinite. This need not be the case. Figure 1, for example, is a complete list of all symmetry transformations of a square. This is an example of a finite group since if you perform one symmetry transformation after another, you will find that what you have just done is described by something already in the list. The group theorist would prefer to denote these symmetry transformations using symbols. Let r stand for the 90 degrees clockwise rotation and let m stand for mirror symmetry $M1$. Using the notational convention $r \cdot r = r^2$ etc., one may describe the group as a whole using the eight words

$$1,\ r,\ r^2,\ r^3,\ m,\ m\cdot r,\ m \cdot r^2,\ m \cdot r^3.$$

Indeed, any other word is equal to one on this list via the three geometric relations

$$r^4 = 1,\ \ m^2 = 1,\ \ m \cdot r \cdot m = r^{-1}.$$

For a more complicated example, one may consider the *Rubik's cube group*. The elements of this group consist of the various cube transformations used to get from one configuration to another. Clearly one transformation followed by another yields yet another transformation, and any transformation may be reversed, whence they satisfy the axioms of a group. In fact the Rubik's cube group has a very complicated algebraic structure, with $43,252,003,274,489,856,000$ elements!

Figure 2: A Rubik's cube.

Simple groups and CFSG

A common way of obtaining new groups from old is by a process known as *extension*. Conversely, groups can be broken down into smaller pieces or *factors*. Certain groups, however, may not be broken down any further and are thus known as *simple groups*. These groups are analogous to prime numbers in the sense that an arbitrary group can be uniquely broken down into its simple factors. Simple groups can therefore be thought of as building blocks, and thus if one seeks to understand all groups, the first step is to understand the simple groups. This leads to what many regard as the greatest mathematical feat of the 20^{th} century: the *classification of finite simple groups*, or CFSG for short. This theorem, the proof of which was completed in 1983, comprises over 500 journal articles and is the work of over 100 authors. The theorem states that every finite simple group is either one of 26 *sporadic groups*, or belongs to one of the following three infinite families:

- *Cyclic groups of prime order*
- *Alternating groups*
- *Groups of Lie type*

The first two on this list are very well understood, which leaves the so called groups of Lie type — in a certain sense by far the most abundant — as the main objects of interest. These can be viewed as the finite analogues and modifications of a swathe of groups which arise naturally as the symmetry groups of multidimensional space. Because of this, they can often be succinctly described as groups of invertible matrices, where composition of symmetry transformations, conveniently, is the same as matrix multiplication.

Representation theory

The technical definition is as follows. A *representation* of a finite group G is a *group homomorphism* into a group of invertible matrices: a *general linear group* $GL_n(\mathbb{C})$. I.e. it is a function

$$\rho : G \longrightarrow GL_n(\mathbb{C}) \ \text{ such that } \ \rho(a \cdot b) = \rho(a) \cdot \rho(b)$$

for all group elements a, b. More intuitively, the representations of a group are the ways in which the group can be realised as the group of symmetries of a multidimensional space. Some reasons for wanting to do this are as follows.

- To gain yet more information about the abstract group itself.
- Many applications in other mathematical fields as well as chemistry.
- Groups offer a description of the symmetries that physical laws seem to obey: representation theory then tells us what the 'possible' physical theories are.

Figure 3: Origami with lots of symmetry!

My research

My PhD research focuses on the representation theory of finite groups of Lie type. For this I use a wide array of mathematical tools coming from algebraic geometry, Lie algebras, linear algebra, homological algebra and group theory. Roughly, I begin by considering certain infinite matrix groups which have a special self-map, a *Frobenius endomorphism* such that the fixed points of this map comprise the finite group of Lie type which I am interested in. These infinite groups are examples of *reductive algebraic groups*, which have a rich geometric structure, that can, due to results of P. Deligne, G. Lusztig and others be exploited in clever ways to study the finite group of Lie type and its representation theory.

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