ELEMENTS OF A THEORY OF ALGEBRAIC THEORIES

For Glynn Winskel at 60
in recognition of his influence
on abstract mathematics

HAPPY

BIRTHDAY

UNPREDICTED IMPACT

Profunctors, Open Maps and Bisimulation Cattani-Winskel

- Non-deterministic processes
  ~ presheaves over $\mathbb{P}$
  ~ transition systems with computation paths based on $\mathbb{P}$

- Bisimulation
  ~ spans of open maps
  (Joyal - Nielsen - Winskel)

- Profunctors ~ cocontinuous functors
  preserve open maps + so bisimulation

AND MORE

TECHNICAL BACKGROUND

- Biequivalence

  Profunctors ~ Presheaf categories, cocontinuous maps

- Ind completion

- Completion under connected colimits

- Families comonad

$\Downarrow$

Kleisli Bicategories

Fiore Gambino Hyland Winskel

Still in preparation

WHAT IS CATEGORY THEORY?

Is it just a language?

What are the big results?

$\left\{\begin{array}{l}\text{accessible} \\ \text{with rich implications}\end{array}\right.$

- Products and equalizers
  $\Longrightarrow$ Limits
  (not obviously rich, but
  Power–Robinson)

- $\mathbb{C} \xhookrightarrow{y} P\mathbb{C} = [\mathbb{C}^{op}, \mathbf{Set}]$ is the closure of $\mathbb{C}$ under colimits
  (accessible or not?)

BASIC PROBLEM

- To understand category theory use 2-categories

$$\rightsquigarrow$$

- To understand algebraic theories use 2-dimensional algebra.

?

PROF IS A BICATEGORY I

$\mathrm{Prof}(A,B) = [A \times B^{op}, \mathrm{Sets}]$

with composition

$$G \circ F(c,a) = \int^{b} G(c,b) \times F(b,a)$$

define unit and associativity 2-cells and check

(which is left to the reader...)

THEN

Prof is biequivalent to a 2-category ....

(with details left to the reader...)

PROF IS A BICATEGORY II

$$
\begin{tikzcd}
 & \mathrm{COCONT} \arrow[d] \\
\mathrm{Cat} \arrow[ur, "P"] \arrow[r, hook] \arrow[ur, phantom, shift right=4, "\dashv"] & \mathrm{CAT}
\end{tikzcd}
$$

Restricted 2-adjunction

$\longmapsto$ Kleisli structure P
(restricted ψ-monad)

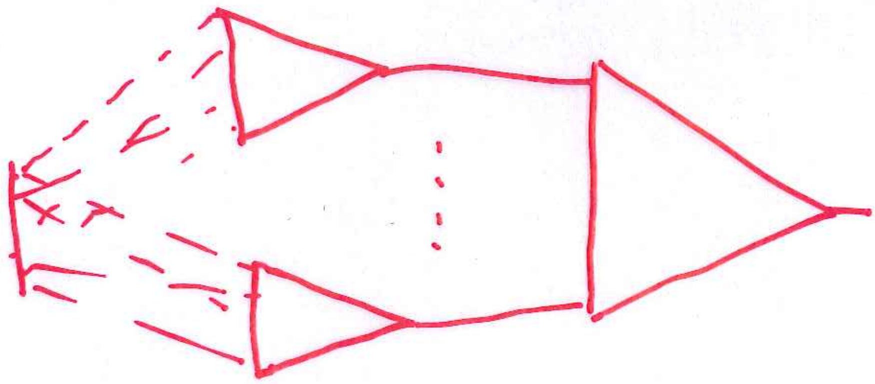
$\longmapsto$ $Kl(P) = \mathrm{Prof}$

BECAUSE it is biequivalent to a 2-category!

ALGEBRAIC THEORIES

$$T : \mathbb{F} \longrightarrow \mathbf{Set}$$

with composition



$$f\left(g_1(\underline{x}_{\sigma 1}), \ldots, g_n(\underline{x}_{\sigma n})\right)$$

E.g. groups

$$x \cdot x^{-1} = e$$

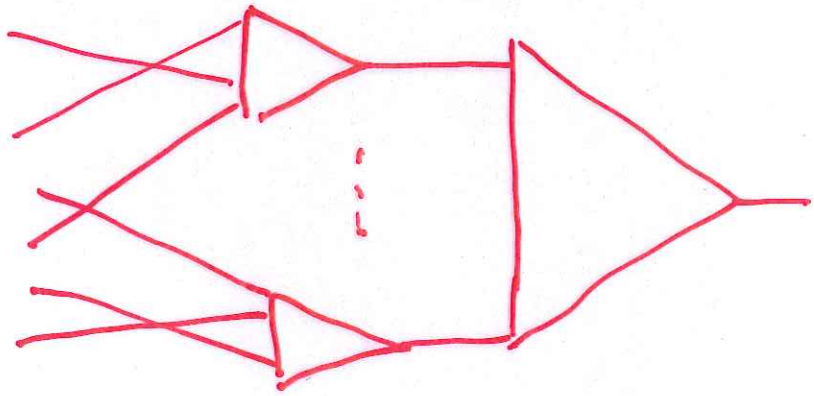
rings

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

SYMMETRIC OPERADS

$\mathcal{O} : \mathbb{B} \longrightarrow \mathrm{Set}$

with composition



$$f(g_1(\underline{x}_1), \ldots, g_n(\underline{x}_n))_\sigma$$

E.g.

commutative monoids

$$xy = yx$$

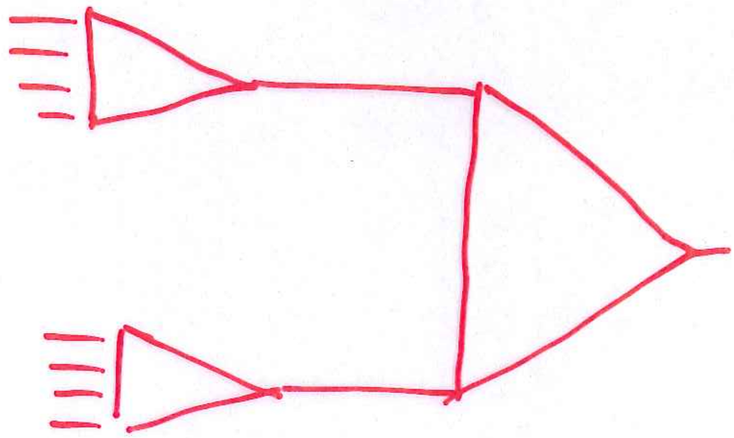
monoids with involution

$$s(xy) = s(y)\,s(x)$$

NON-SYMMETRIC OPERADS

$\mathcal{O} : \mathbb{N} \longrightarrow \mathrm{Set}$

with composition



$f(g_1(\underline{x}_1), \ldots g_n(\underline{x}_n))$

E.g. monoids

$x.(y.z) = (x.y).z$

IDEA

Style of composition

$\sim$ 2-monads on Cat
(with extension to Prof)

Algebraic theories

$\sim$ categories with products

Symmetric operads

$\sim$ symmetric monoidal categories

Non-symmetric operads

$\sim$ monoidal categories

EXTENDING TO PROF

S a 2-monad on Cat

$$\frac{A \nrightarrow B \;=\; A \longrightarrow PB}{SA \nrightarrow SB \;=\; SA \longrightarrow SPB \longrightarrow PSB}$$

distributivity

25

LIFTING

$$\begin{array}{c} S\text{-Alg} \circlearrowleft P \\ \downarrow \\ \text{Cat} \circlearrowleft P \end{array}$$

- If A an S-algebra, so is PA so that $A \xrightarrow{y} PA$ "is" a ψ-map
- If $f : A \longrightarrow PB$ is a ψ-map then so is $f^{+} : PA \longrightarrow PB$

(left Kan extension)

NON-EXAMPLES / EXAMPLES

NO

Biproducts

Coproducts (finite)

Equalizers

YES

Products

Symmetric monoidal

Monoidal

Cattani - Weiskel

$\Longrightarrow$ Conjecture characterizing the extending theories of finite limits

PRELIMINARY DEFINITION

S extended monad on Prof

An S-algebraic theory is a monad in the Kleisli bicategory

$\mathrm{Kl}(S)$

i.e.
$$\begin{tikzcd}
 & SA \arrow[dr, "M^\#"] \arrow[d, phantom, "\Downarrow"] & \\
A \arrow[ur, "M"] \arrow[rr, "M"] \arrow[rr, bend right=50, "\eta_*"'] & {} \arrow[d, phantom, "\Uparrow"] & SA \\
 & {} &
\end{tikzcd}$$

satisfying axioms.

FREE CONSTRUCTION

Each S-algebra $SA \xrightarrow{a} A$

gives an S-algebraic theory

$$A \overset{a^*}{\longrightarrow\!\!\!\!\!\!\!+\;\;} SA$$

This has a left adjoint

the free S-algebra generated

by $A \overset{M}{\longrightarrow\!\!\!\!\!\!\!+\;\;} SA$.

WARNING

For S = Products, S-algebras

and S-algebraic theories are

equivalent.

FALSE for

S = Symmetric monoidal

Monoidal

ENCOURAGING THEORY

$$
\begin{tikzcd}
\text{Coloured Algebraic theories} \arrow[d, shift left=3, "\text{forgetful}"] \\
\text{Coloured Symmetric operads} \arrow[u, shift left=3, "\text{free}"] \arrow[u, phantom, "\dashv"] \arrow[d, shift left=3, "\text{forgetful}"] \\
\text{Coloured Non-symmetric operads} \arrow[u, shift left=3, "\text{free}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

PROBLEMS

Names

Symmetry

Rigid operads

Shuffle operads

RIGID OPERADS

A symmetric operad $\mathcal{O}$ is rigid just when all stabilizers in $\mathcal{O}(n)$ are trivial.

(So all non-symmetric operads but more ....)

~ Polynomial monads

~ Hermida - Makkai - Power

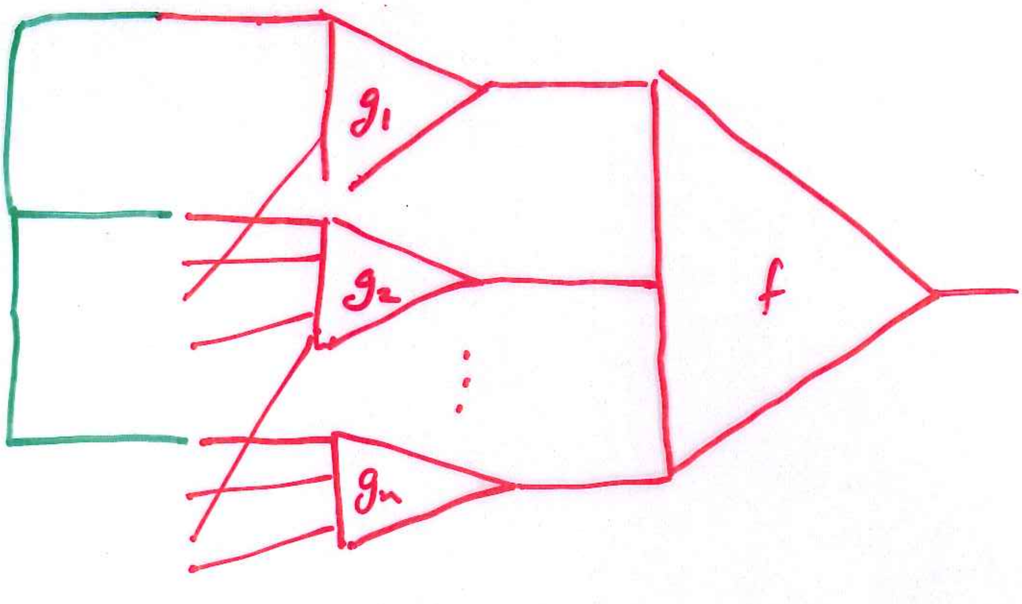
WARNING : Being rigid is undecidable.

SHUFFLE OPERADS

(Vladimir Dotsenko)

Shuffle composition

(on $\mathcal{O} : \mathbb{N} \longrightarrow \mathrm{Sets}$)



CONCLUSION

Why are there more questions than answers?

Still so much to do!

So

HAPPY BIRTHDAY

and

MANY HAPPY RETURNS