45 years on: Some Abstract Mathematics of Eilenberg

Martin Hyland
(Cambridge)

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Two case studies in conceptual foundations:

Samuel Eilenberg and Calvin C. Elgot.
Recursiveness (Academic Press 1970)

Samuel Eilenberg and Jesse B. Wright
Automata in General Algebras
(AMS Summer Meeting 1967)

RECURSIVENESS

Aim to characterize the recursive functions, partial functions, relations on $\mathbb{N}$ or generally $W = L^*$ (with L finite).

Relational setting

Consider subcategories $\mathcal{C}$ of **Rel** with objects the X^n for some X

Basic assumptions

$$\mathbb{F}^{op} \longrightarrow \mathcal{C}$$
$$n \longmapsto X^n \quad \text{functorial}$$

(so for functions we have Lawvere theories)

$1, \times$ give monoidal structure

(so relations in Carboni-Walters spirit)

FREE MONOIDS

For $W = L^*$ (L finite + assumed $\neq \emptyset$)

E+E give structure

$$1 \longrightarrow W \qquad * \longmapsto e$$

$$L \times W \longrightarrow W \qquad l, u \longmapsto l.u$$

$$W \times L \longrightarrow W \qquad u, l \longmapsto u.l$$

and iterations

$$\frac{L \times X \longrightarrow X}{W \times X \longrightarrow X} \qquad \frac{X \times L \longrightarrow X}{X \times W \longrightarrow X}$$

all functions. This generates

the Primitive Recursive Functions

(N.B. One action + iteration suffices.)

Weak but sufficient to give

parametrised primitive recursion.

CODING

I $\quad \mathbb{N}^2 \xrightarrow{\cong} \mathbb{N}$

$$x, y \longmapsto x + \tfrac{1}{2}(x+y)(x+y+1)$$

or

$$x, y \longmapsto 2^x(2y+1) - 1$$

II $\quad W = L^* \cong \mathbb{N} = 1^*$

by the lexicographic ordering

Hence we have isomorphisms

$$\gamma : L^* \xrightarrow{\cong} M^*$$

which are primitive recursive in the sense that for $l \in L$ $u \in L^*$

$$\gamma(u) \longmapsto \gamma(l.u)$$

is primitive recursive. $M^* \longrightarrow M^*$

So categories are all isomorphic.

THEORY OF PRIMITIVE RECURSION

Let **PRIM** be the initial category with (strict) finite products and a parametrized natural number object $1 \xrightarrow{0} \mathbb{N} \xleftarrow{s} \mathbb{N}$

Syntactic presentation

objects are $1 = \mathbb{N}^0, \mathbb{N}^1, \mathbb{N}^2, \ldots,$

maps are primitive recursive expressions up to provable equality (i.e. quantifier free induction, i.e. equality from the universal property of the nno).

It is a Lawvere theory (for logicians the theory of primitive recursive arithmetic)

Exercise $x + y = y + x$

THEORY vs MODEL

Since **Sets** has a parametrised nno we have $\mathbf{PRIM} \longrightarrow \mathbf{Sets}$ and the primitive recursive functions are the image. What's the difference?

PRIM does not have enough points!

- Equality of primitive recursive expressions is semi decidable (from the recursive theory)

Lemma The points of $\mathbb{N}$ in PRIM are just the natural numbers $1 \xrightarrow{n} \mathbb{N}$

- So if PRIM had enough points then inequality of primitive recursive expressions would be semi-decidable: $f \neq g \Leftrightarrow \exists n\, f(n) \neq g(n)$

But then equality would be decidable contradicting basic recursion theory.

AXIOMATISATION

PRIM is the natural setting for the arguments and coding given by E+E for primitive recursive functions.

E.g. each free category with products containing a parametrized list object on a non-ϕ finite set is coded in PRIM and all these categories are isomorphic

QUESTION What happens when we pursue E+E's agenda using PRIM rather than its interpretation in Sets ?

RECURSIVE RELATIONS

(title of E+E Chapter IV)

Category of relations — logical functions
— × functorial

inverse
$$\frac{X \xrightarrow{r} Y}{Y \xrightarrow{r^{\circ}} X}$$

free monoid
$$1 \longrightarrow W$$
$$L \times W \longrightarrow W$$

recursion
$$\frac{X \xrightarrow{r} X}{X \xrightarrow{r^{*}} X}$$
where

$$r^* = I + r + r^2 + r^3 \dots$$

Proposition The functions on any sets are closed under primitive recursion.

MAIN THEOREM

Suppose $\mathcal{C}$ is a category of functions closed under primitive recursion; then $\mathcal{C}^{-1}\mathcal{C}$ is a category of recursive relations.

Here $\mathcal{C}^{-1}\mathcal{C}(X,Y)$ consists of $\emptyset$ plus all relations which factorize as

$$X \xrightarrow{f^o} Z \xrightarrow{g} Y$$

with $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $\mathcal{C}$

(Better formulation would use eg decidable subsets D determined by $\mathcal{C}$ and consider

$$X \leftarrow D \rightarrow Y \quad)$$

THEORY OF RECURSIVE RELATIONS

Outline

Carboni-Walters - order enriched R

- left adjoints give $Map(R)$
- symmetric monoidal $1, \times$
- map valued lax transformations

$$A \to A \times A$$
$$A \to I$$

with axioms making $1, \times$ products on $Map(R)$

- adjunction $\dfrac{A \times B \nrightarrow C}{A \nrightarrow B \times C}$

E+E style recursion
$$1 \longrightarrow W$$
$$L \times W \longrightarrow W$$

$$\dfrac{X \overset{r}{\nrightarrow} X}{X \overset{r^*}{\nrightarrow} X}$$

Proposition (E+E) $Map(R)$ is closed under iteration

REGULAR CATEGORIES

For an abstract version of the Main Theorem we need to construct categories of relations and for that we need images ~ regular categories

$$\begin{tikzcd}
\text{Categories of relations} \arrow[dr] & & \text{Regular categories} \arrow[ll] \\
& \text{Regular hyperdoctrines} \arrow[ur] &
\end{tikzcd}$$

Try to do this with a regular hyperdoctrine on PRIM

THE FREE ARITHMETIC UNIVERSE
(Joyal)

Take the regular hyperdoctrine / PRIM

where $R(A) =$ the set of formal $\exists x \in X . P(a,x)$
with P a primitive recursive predicate

with the simple preorder

$$\exists x \in X . P(a,x) \vdash \exists y \in Y . Q(a,y)$$

iff for some $\pi : A \times X \longrightarrow Y \in$ PRIM

$$P(a,x) \leq Q(a, \pi(a,x))$$

Then PRIM[R] is Joyal's free arithmetic universe; for the regular category PRIM^{ass}[R] which we seek has disjoint coproducts and parametrized lists.

OUTCOME

Take **Rel(R)** the category of relations of the regular $\mathbf{PRIM}^{ass}[R]$ (where we could restrict the objects to make the analogy more exact)

Fact **Rel(R)** is a category of recursive relations.

However there is a mismatch!

For E+E the maps in the category of relations coming from primitive recursive functions are all recursive functions.

For us the maps in **Rel(R)** are just the maps/morphisms of **PRIM**

MORAL LESSON

The chosen preorder is what determines the regular category, the category of relations + finally the maps in it.

TRUE ARITHMETIC

We get the category of recursive functions as in E+E.

PROVABILITY IN PA

We get the category of provably recursive function expressions $\boxed{\text{up to provable equality}}$

This is a natural extension of the traditional proof theory focus.

AUTOMATA IN GENERAL ALGEBRAS

Setting

Algebraic theories and their algebras

First half of paper is an outline of the basic material on Lawvere theories.

Key point

The free algebraic theory on a given signature

E&W give a characterization in terms of the degree of elements of the theory: it counts the maximum depth of substitutions needed to construct the term.

THE DEFINITION

Given a theory T, a $\boxed{T\text{-automaton}}$ is a finite T-algebra A equipped with some subset $A_f \subseteq A$ of final states.

Given such a T-automaton A we have a unique map

$$\zeta : T(0) \longrightarrow A$$

from the initial T-algebra.

The $\boxed{\text{behaviour}}$ of A is the subset

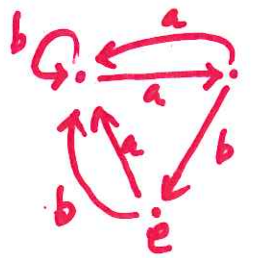
$$\zeta^{-1}(A_f) \subseteq T(0)$$

of the initial algebra.

EXPLANATION

Regard the monoid W of words in a finite alphabet L as a theory with a constant e and unary operators $l(-)$ for each $l \in L$, namely the free theory so generated. The initial algebra is isomorphic to W.

Typical automaton looks like a finite algebra



plus a choice of final states.

So it is a deterministic and totally defined automaton with initial state e

EXTENDING THEORIES

When does it make sense to talk of a relational T-algebra?

$$
\begin{tikzcd}
\mathrm{T\text{-}alg} \circlearrowleft \hat{P} \arrow[d, "\text{lift}"] & \Leftrightarrow & \mathrm{Kl}(P) = \mathrm{Rel} \circlearrowleft \tilde{T} \\
\mathrm{Set} \circlearrowleft P \arrow[dr, phantom, "\Leftrightarrow" sloped] & & \mathrm{Set} \circlearrowleft T \arrow[u, "\text{extension}"'] \arrow[dl, phantom, "\Leftrightarrow" sloped] \\
& TP \xrightarrow{\lambda} PT &
\end{tikzcd}
$$

distributive law

Any operadic theory has a canonical distributive law – so in particular any free theory.

RELATIONAL ALGEBRA

$\hat{P}$-alg $= PT$-alg	T-algebras in V-lattices (with T structure V-preserving)
$\tilde{T}$-alg	T-algebras in free V-lattices = Rel
$Kl(\hat{P})$	T-algebras in Rel which result from applying $\hat{P}$ to a T-algebra in Sets
$Kl(\tilde{T}) = Kl(PT)$	T-algebras in Rel which come from a free T-algebra in Sets

AN ADJUNCTION

The category $\widetilde{T}$-Alg of relational algebras can be presented with objects

$$TA \relbar\joinrel\mapstochar\joinrel\rightarrow A$$

i.e. with a relational structure map

Then we have

$$\begin{array}{ccc} TX \to X \to PX & \widetilde{T}\text{-Alg} & TX \to PX \\ \uparrow & \dashv & \downarrow \\ TX \to X & T\text{-Alg} & TPX \xrightarrow[\lambda]{} PTX \to P^2X \xrightarrow[\mu]{} PX \end{array}$$

an adjunction. This explains

E+W ... the initial T-algebra is also initial in the larger category of relational T-algebras.

RELATIONAL AUTOMATA

Finite relational T-algebras

$$TA \nrightarrow A$$

with set $A_f \subseteq A$.

We have unique $T0 \overset{\xi}{\nrightarrow} A$ and the behaviour is

$$\{k \in T0 \mid \xi(k) \cap A_f \neq \phi\}$$

(Correct for the leading example.)

Consider the ordinary (deterministic) T-automaton PA with

$$(PA)_f = \{X \subseteq A \mid X \cap A_f \neq \phi\}$$

Then A and PA have the same behaviour.

The behaviours of deterministic and non-deterministic automata the same.

FORMAL SUMS

Consider formal sums

$$\underline{t}(\underline{x}) = t_1(\underline{x}) + \cdots + t_k(\underline{x})$$

of elements of $T(n)$ (terms with n free variables).

If A is a relational algebra then $\underline{t}(n)$ acts

$$PA^n \longrightarrow PA$$

$$X_1, \ldots, X_n \longmapsto \bigcup \left\{ t_i(a_1, \ldots, a_n) \;\middle|\; \begin{array}{c} 1 \le i \le k \\ a_j \in X_j \end{array} \right\}$$

If τ an n-tuple of such then we get

$$PA^n \xrightarrow{\tau} PA^n$$

Since τ is continuous we can take $\overline{\tau}_A \in PA^n$ the least fixed point.

Evidently for $\gamma : T(0) \longrightarrow A$ we have

$$\gamma : \overline{\tau}_{T(0)} \longmapsto \overline{\tau}_A$$

GENERALIZED REGULAR EXPRESSIONS

These are the subsets (infinite sums) which appear as a coordinate of the fixed point of some $T : P(T(0))^n \longrightarrow P(T(0))^n$

Sanity check (in the leading example)

$$(e + a(x) + b(y),\ c(x) + d(y)) : P(W)^2 \longrightarrow P(W)^2$$

Fixed point iteration:

$$(\phi, \phi) \longmapsto (e, \phi) \longmapsto (e + a(e),\ c(e))$$

$$\longmapsto (e + a(e) + a^2(e) + bc(e),\ c(e) + ca(e) + dc(e))$$

Regular algebra computation of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} e \\ 0 \end{pmatrix} :$$

$$\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} e \\ 0 \end{pmatrix} = \begin{pmatrix} e \\ 0 \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \begin{pmatrix} e \\ 0 \end{pmatrix} = \begin{pmatrix} a^2 + bc & - \\ ca + dc & - \end{pmatrix}$$

matches!

BASIC THEOREMS

① Reduction to formal sums of generating (i.e. degree 1) terms.

② Assume T free on a finite signature

Let A be a finite relational algebra with $|A| = \{1, \dots, n\}$. For each i consider the formal sum of generating terms

$$t(x_{f(1)}, \dots x_{f(k)}) \qquad \begin{pmatrix} f : k \to n \\ t \in T(k) \end{pmatrix}$$

such that in A

$$t(f(1), \dots f(k)) = i.$$

(This exactly determines the relational algebra structure.)

The fixed point of the resulting τ is $(\zeta^{-1}(1), \dots, \zeta^{-1}(n))$ so gives the behaviour for all choices of A_f.

MAIN THEOREM

(Mezei - Wright)

For a free theory T on a finite base the behaviours (recognizable sets) and regular sets coincide.

- The behaviours are $\xi^{-1}(A_f)$ so unions of regular $\xi^{-1}(i)$: and regulars closed under union
- Conversely a regular is exactly a behaviour.

General form of the classification for usual automata in terms of regular languages.

REFORMULATION

Address the issue that the E+W theory is highly dependent on the constants (Cf the leading example)

A T-automaton with n inputs and m outputs is a finite (relational) T-algebra A with a diagram

$$\begin{tikzcd} T(n) \arrow[r, "\shortmid"{marking}] & A \\ & T(m) \arrow[u, "\shortmid"{marking}] \end{tikzcd}$$

a surjection

The composition (multiplication) of behaviours becomes (again) evident

$$\begin{tikzcd} T(n) \arrow[r, "\shortmid"{marking}] & A \arrow[r] \arrow[dr, phantom, "\lrcorner"] & C \\ & T(m) \arrow[u, "\shortmid"{marking}] \arrow[r, "\shortmid"{marking}] & B \arrow[u] \\ & & T(k) \arrow[u, "\shortmid"{marking}] \end{tikzcd}$$

(But no identities!)

CONCLUSION

E+W: It is clear from this introduction that this paper contains nothing that is essentially new, except perhaps for a point of view.

Recursion, automata theory are areas of mathematics in which the conceptual foundations provided by category theory are felt to play no role.

My two examples show Eilenberg opposing that 45 years ago.

I draw attention to that and the possibility of continuing the struggle better to understand such areas today.