OPERADS AND ALGEBRAIC THEORIES

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PLAN

(Foundations with a glance at applications)

INFORMAL REFLECTIONS

KLEISLI BICATEGORIES

A THEORY OF THEORIES

MIXED THEORIES (worked example)

EXEMPLARY APPLICATION

M. Abadi + G. Plotkin:
A model of cooperative threads.

Imperative programs execute w/o interruption until they explicitly yield control (or terminate).

Based on a calculus of merging.

What is that?

DENDRIFORM ALGEBRAS

$$\begin{array}{c}\lhd\\ \rhd\end{array} : E \otimes E \longrightarrow E$$

$$(a \lhd b) \lhd c = a \lhd (b \lhd c) + a \lhd (b \rhd c)$$

$$(a \rhd b) \lhd c = a \rhd (b \lhd c)$$

$$a \rhd (b \rhd c) = (a \rhd b) \rhd c + (a \lhd b) \rhd c$$

Example: the Shuffle Algebra

$a \lhd b$ is merge a and b with first element from a.

DENDRIFORM TENSORS

J. Laird. A categorical semantics of higher order store.

introduced $A \lhd B$ and $A \rhd B$ with

$$A \otimes B = A \lhd B \oplus A \rhd B$$

P. Clairambault.

Laird's games model $\sim$ Conway games

MORAL

Combinatorics of operads

$\sim$

Combinatorics of

data handling, data flow.

[Just one example of Category
Theory as a tool in IT.]

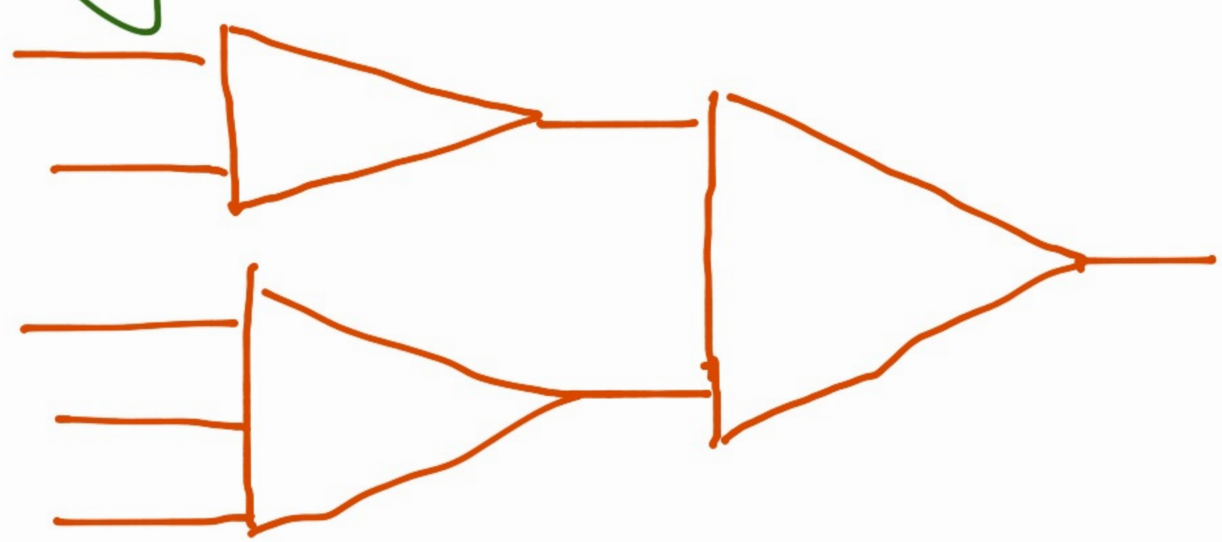
NON-SYMMETRIC OPERADS

$\mathcal{O}(n)$ with $id \in \mathcal{O}(1)$

$$\mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \mathcal{O}(m_n) \longrightarrow \mathcal{O}(m_1 + \cdots + m_n)$$

satisfying unit and associativity

PICTURE



NON-SYMMETRIC LINEAR THEORIES

Equations with the same variables appearing once + in the same order on either side.

EXAMPLES

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

$$1 \cdot a = a = a \cdot 1$$

$$a \dashv (b \dashv c) = a \dashv (b \vdash c)$$

$$\vdots$$

Dendroidal algebras are a linear version

SYMMETRIC OPERADS

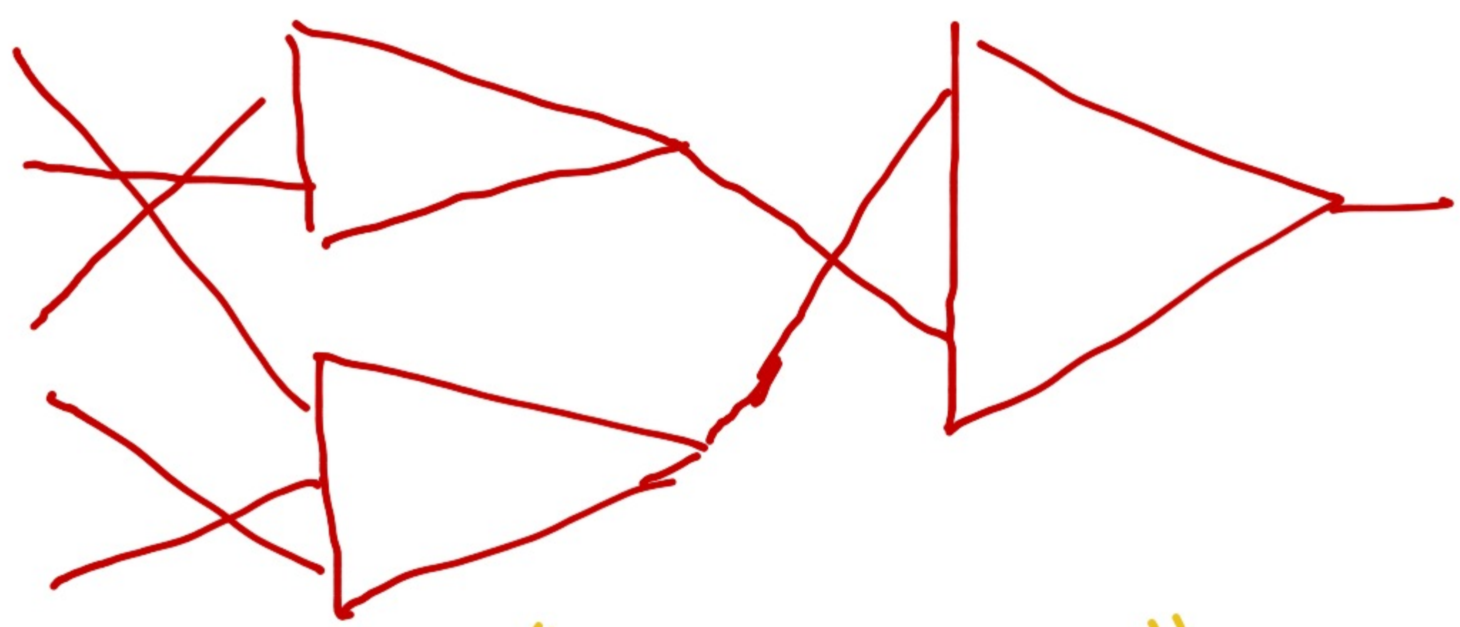
$\mathcal{O}(n) \times S_n \longrightarrow \mathcal{O}(n)$ with $id \in \mathcal{O}(1)$

$$\mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n)$$

$$\longrightarrow \mathcal{O}(m_1 + \cdots + m_n)$$

satisfying unit associativity and action conditions

PICTURE



Baez-Dolan: "Combing"

LINEAR (REGULAR) THEORIES

Equations with the same variables appearing once on either side.

EXAMPLES

$$a.b = b.a$$

$$\overline{a.b} = \overline{b}.\overline{a}$$

Linear cases

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

$$\vdots$$

ALGEBRAIC THEORIES I

$\mathcal{O} : \mathbb{F} \longrightarrow \mathrm{Set}$ (i.e. $\mathcal{O}(n)$ with right action by $\mathbb{F}^{op}$) $\mathrm{id} \in \mathcal{O}(1)$

$$\mathcal{O}(n) \times \mathcal{O}(m_1) \cdots \mathcal{O}(m_n) \longrightarrow \mathcal{O}(m_1 + \cdots + m_n)$$

satisfying unit, associativity and action conditions

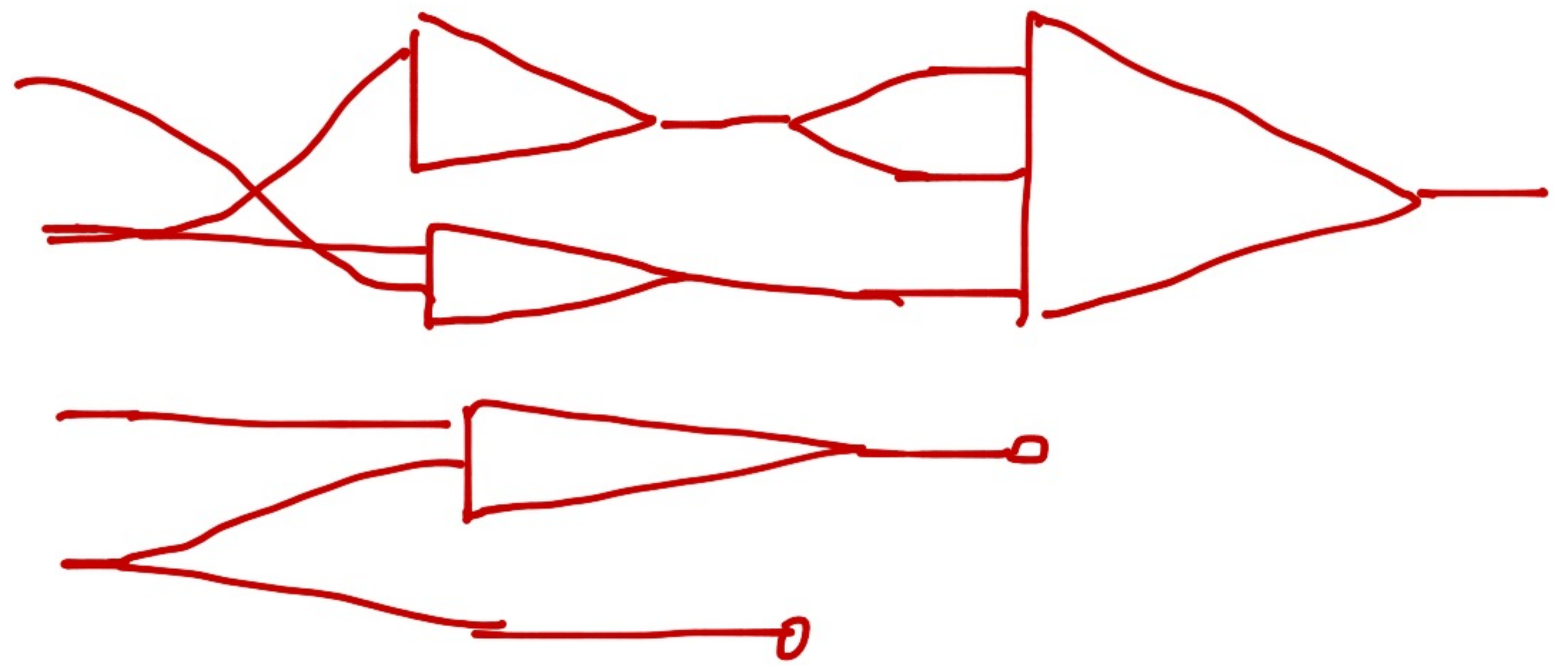
(Operadic presentation $\rightsquigarrow$

$$\mathcal{O}(n) \times \mathcal{O}(m)^n \longrightarrow \mathcal{O}(m) \rightsquigarrow$$

abstract clones.)

ALGEBRAIC THEORIES II

PICTURE



EXAMPLES

$$x \cdot x^{-1} = 1$$

$$x \cdot (y+z) = x \cdot y + x \cdot z$$

(N.B. + with completely different structures :-)

BICATEGORY OF PROFUNCTORS

(Distributors)

Prof has 1-cells $A \overset{M}{\not\longrightarrow} B$ as

$$\frac{A \times B^{op} \longrightarrow \mathrm{Set}}{A \longrightarrow PB}$$

so is the Kleisli bicategory for the presheaf (restricted, pseudo) monad P.

(Why is it a bicategory?)

MAIN IDEA

(with Fiore, Gambino, Winskel)

Seek 2-monads S on $\underline{\mathrm{Cat}}$ which extend to (pseudo monads) on $\underline{\mathrm{Prof}}$.

$$\begin{tikzcd}
\underline{\mathrm{Prof}} \arrow[loop right, "S"] \\
\underline{\mathrm{Cat}} \arrow[u] \arrow[loop right, "S"]
\end{tikzcd}$$

Then consider $Kl(S/\underline{\mathrm{Prof}}) \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \underline{\mathrm{Prof}}$

(a biadjunction)

EXTENSION / LIFTING

(Categorical level)

P, S monads on $\mathcal{C}$

$$\begin{array}{ccccc} S\text{-}Alg \circlearrowleft P & & & & Kl(P) \circlearrowleft S \\ \text{lifting} \Big\downarrow & \sim & \begin{array}{c} \lambda : SP \longrightarrow PS \\ \text{distributive law} \end{array} & \sim & \Big\uparrow \text{extension} \\ \mathcal{C} \circlearrowleft P & & & & \mathcal{C} \circlearrowleft S \end{array}$$

And then PS is a monad

etc etc etc

RELATIONAL ALGEBRA

(Baby example)

Which monads extend from Sets to $\mathrm{Rel} = \mathrm{Kl}(P)$ (respecting the order)?

Power set

$$\begin{array}{c} \mathrm{Rel} \circlearrowleft S \\ \uparrow \\ \mathrm{Set} \circlearrowleft S \end{array}$$

Confusing answer: certainly all operadic theories but many more.

LIFTING PRESHEAVES

from $\underline{Cat}$ to $S\text{-}\underline{Alg}$

Need (choice of structure so that)

- If A is a T-algebra then PA is a (pseudo) T-algebra and $A \xrightarrow[y_A]{} PA$ is a (pseudo) map of such.

- If $f : A \longrightarrow PB$ is a (pseudo) map then so is the left Kan extension $f^{\#} : PA \longrightarrow PB$ (and the composite $f^{\#} \cdot y_A$ is isomorphic to f.

EXAMPLES & NON-EXAMPLES

	Biproducts
Terminal object	Initial object
Finite products	Finite coproducts
Finite limits	Equalizers
(Symmetric) monoidal	
Endofunctor	
Factorization	

KLEISLI BICATEGORIES

(for 2-monads which extend)

1-cells $A \xrightarrow{M} SB$

Identities $A \xrightarrow{\eta_*} SA$

Composition

$$\frac{A \xrightarrow{M} SB \quad B \xrightarrow{N} SC}{A \xrightarrow{M} SB \xrightarrow{SN} S^2C \xrightarrow{\mu_*} SC}$$

USES

General

Concurrency (Cattani-Winskel: after Joyal-Nielsen-Winskel)

Variable binding (Edinburgh)

Theory of theories (this talk)

Special

Species: 2D Diff λ-calculus

Mixed theories (this talk)

THEORY OF THEORIES

Preliminary version (w. Power, Garner)

Take S an extended monad: an S-theory (S-multicategory) is a (+) monad $A \overset{M}{\longrightarrow\!\!\!\!\!\!\!+} \;\; SA$ in the Kleisli bicategory $Kl(S/\underline{Prof})$.

(+) discrete category of objects or (better) normal monad.

"Evident" notion of map of such.

LEADING EXAMPLES

(multicoloured)

$S =$ monoidal — non-symmetric operads

$S =$ symmetric monoidal — symmetric operads

$S =$ products — algebraic theories

S-ALGEBRAS

Each S-Algebra $SA \xrightarrow{a} A$ induces an S-Theory $A \overset{a^*}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} SA$.

There is a left biadjoint, the free S-Algebra generated by $A \xrightarrow[M]{+} SA$.

Construction $SA \xrightarrow{SM} S^2A \xrightarrow{\mu_A} SA$ is a monad: take its Kleisli object in $S\text{-}\mathrm{Prof} \subseteq S\text{-}\mathrm{Prof}_{\mathrm{lax}}$

SPECIAL CASE

If S preserves b.o.o. functors then it is sufficient to take the Kleisli in $\underline{\mathrm{Prof}}$.

Examples (one object)

S = monoidal	monoidal category from non-symmetric operad
S = symmetric monoidal	PROP from symmetric operad
S = products	Lawvere theory from algebraic theory

DETOUR

(Blackwell – Kelly – Power)

$$\begin{tikzcd} T\text{-}\mathrm{Alg}_{st} \arrow[rr, hook] & \perp & T\text{-}\mathrm{Alg}_{lax} \arrow[ll, bend right, "(-)^+"'] \end{tikzcd}$$

gives a command: for extended monads S, A^+ is the Kleisli in

$$\begin{tikzcd} SA \arrow[d, "Sa^*"'] \arrow[dr] & \\ S^2A \arrow[d, "\mu_*"'] & A^+ \\ SA \arrow[ur] & \end{tikzcd}$$

COALGEBRAS

The free S-Alg generated by $A \overset{\mu}{\longmapsto} SA$ is naturally a $(-)^{+}$-coalgebra.

$$\text{S-Alg}_{\text{lex}} \hookrightarrow \text{S-Theories} \hookrightarrow (\ )^{+}\text{-Coalg}$$

≀

Free $(-)^{+}$-Coalg

≀

S-Theory generated by S-Algebra

An equivalence in the standard examples

(Folklore)

DELICATE POINT

S-Theories $\hookrightarrow$ $(-)^+$-Cralg

is an equivalence when S preserves b.o.o. functors and certain limits

Question. In general is $(-)^+$-Cralg the right general notion of algebraic theory?

END OF DETOUR

PRODUCTS : SPECIAL FEATURE

The 2-monad for products is co-lax idempotent.

So lax maps are pseudo and $(-)^{\dagger} = (-)'$.

Then each S-algebra $A \simeq A'$ which is a free $(-)'$ coalgebra.

$\therefore$ Product theories $\simeq$ Categories with $\times$s

(Lawvere theories)

Notoriously false in other cases.

CHANGE OF BASE

Let $\lambda : S \longrightarrow T$ be a map of extended 2-monads. We get

$$\mathrm{Kl}(PS)(A,B) \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathrm{Kl}(PT)(A,B)$$

$$C \underset{N}{\nrightarrow} TD \overset{\lambda^*}{\nrightarrow} SD \longleftarrow\!\shortmid \; C \underset{N}{\nrightarrow} TD$$

$$A \underset{M}{\nrightarrow} SB \longmapsto A \underset{M}{\nrightarrow} SB \underset{\lambda_*}{\nrightarrow} TB$$

The left adjoint $\lambda_* \circ -$ gives a homomorphism of bicategories with right adjoint $\lambda^* \circ -$ a morphism.

(Day phenomenon.)

FREE & FORGETFUL

Map $\lambda : S \to T$ of extended 2-monads

From λ^*_0 – get forgetful from T-theories to S-theories.

From λ_{*0} – get free T-theory generated by an S-theory.

Examples (one object case)

$$\text{Non-sym Operads} \underset{\text{free}}{\overset{\text{forget}}{\rightleftarrows}} \text{Sym. Operads}$$

$$\text{Sym Operads} \underset{\text{free}}{\overset{\text{forget}}{\rightleftarrows}} \text{Alg theories}$$

WARNING I

(Leinster)

The free

Non. Sym Operads $\longrightarrow$ Sym. Operads

does not reflect isomorphisms.

So saying

a certain symmetric operad is in fact non-symmetric

is not quite precise.

LINEAR/NON-LINEAR OPERADS

(w. Tasson)

Motivation

From $f : U \to V$ non-linear we get

$Df : U, U \to$ with $Df_u(h)$ non-linear in u but linear in h.

So seek a primitive arena in which to distinguish linear and non-linear maps

(Foundations of Differential λ-calculus
Ehrhard – Regnier)

MIXED MONAD I

Take $S \xrightarrow{\lambda} T$ map of 2-monads

For each A take the lax colimit

$$\begin{tikzcd}
SA \arrow[d, "\lambda_A"'] \arrow[dr] & \\
TA \arrow[r] \arrow[ur, phantom, "\Nearrow"] & MA
\end{tikzcd}$$

in the 2-category of S-Algebras and strict maps.

MIXED MONADS II

The lax colimit MA can be given the structure of a 2-monad.

We have natural

$$\begin{tikzcd}
S \arrow[d, "\lambda"'] \arrow[dr] \arrow[drr, bend left] & & \\
T \arrow[r, phantom, shift left=4, "\Uparrow"] \arrow[r] \arrow[rr, bend right] & M \arrow[r] & T
\end{tikzcd}$$

with all these maps of 2-monads

MIXED ALGEBRAS

An M-algebra is an S-algebra equipped with a retraction onto a T-algebra.

Hence (there is work to do)

If presheaves $\mathcal{P}$ lifts to ψ-S-Alg and ψ-T-Alg then it lifts to ψ-M-Alg; so there will be a Kleisli bicategory

$$\mathrm{Kl}(\mathcal{P}M).$$

MIXED OPERADS

Special case

S = symmetric monoidal

T = finite products

Then an M-operad is essentially

$\mathcal{O}(a;n)$ 'functions' with

a linear arguments (~ symmetric operads)

n non-linear arguments (~ products)

and a natural

$$\mathcal{O}(a+1;n) \longrightarrow \mathcal{O}(a;n+1).$$

THEORIES w. DIFFERENTIATION

Mixed operads $\mathcal{O}(a,n)$ equipped with a natural

$$\mathcal{O}(a;n+1) \xrightarrow{D} \mathcal{O}(a+1;n+1)$$

satisfying

- the derivative of a linear map is itself
- the chain rule.

PURE λ-CALCULUS

Old idea

An interpretation of the λ-calculus is a semi-closed algebraic theory $\mathcal{L}$.

New point

Presheaves on $\mathcal{L}$ ($\mathcal{L}$-modules)

$$X(n) \times \mathcal{L}(m)^n \longrightarrow X(m)$$

is a presheaf category coding the fundamental theory of the λ-calculus.

H. Classical λ-calculus in modern dress.

CURRENT PROGRAMME

Use the same approach as for λ-calculus in the case of theories with differentiation.

Hope for fresh light on the categorical foundations of differential λ-calculus.

For another time!

CONCLUSION

This talk presented an eclectic selection of aspects of the foundations of operads.

Many more aspects scheduled through the course of the Journées.