FORMAL TOPOLOGY

AND

COMMUTATIVE ALGEBRA

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3WFTop

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CONSTRUCTIVE

MATHEMATICS

Two influences

Martin-Löf Type theory as a compelling, elegant setting.

Bishop Constructive mathematics looks natural / normal.

What should constructive
Commutative Algebra
Algebraic Geometry
look like?

TOPOS / LOCALE

THEORY ANALYSIS

(Joyal, Mulvey)

Constructive variants of classical results in terms of locales.

(Classifying toposes.)

CONSTRUCTIVE CONTENT

of classical commutative algebra (& much else)

(Coquand, Lombardi et al)

Extract the algebraic manipulations etc hidden in classical arguments using ideal objects, Zorn's Lemma etc.

Hilbert's Programme

Ideal elements as eliminable in proofs of concrete statements.

CONSTRUCTIVE COMMUTATIVE ALGEBRA

Should be conducted in a predicative constructive setting

(Martin-Löf)

· Ideal elements should appear so it looks normal (Bishop!)

IDEA Use some form of formal topology. Ideal elements as the 'points' of a formal space.

But then how to argue?

Analogy with ASD

OLD TECHNIQUE

(Joyal, Moerdijk, Wraith)

Extend a topos
(adding ideal elements,
making it Boolean ....);
prove something in the
extended (simpler)
situation;
derive information about
the original topos (descent).

DREAM

Explain mod. techniques via 'the' Logic of all forcing extensions.

Example: Cardinal collapse

One can extend to make any set countable and (Joyal) via an open surjection.

So anything might as well be countable.

(So the Dream logic is weak: certainly predicative.)

WHAT IS A PRIME IDEAL?

Constructively an 'element' of some 'formal space'.

BUT WHICH?

ZARISKI SPECTRUM

(Joyal)

$$\vdash D(1)$$

Multiplicative set

$$D(x), D(y) \vdash D(xy)$$

$$D(xy) \vdash D(x)$$

Saturation

$$D(xy) \vdash D(y)$$

$$D(0) \vdash$$

Prime

$$D(x+y) \vdash D(x), D(y)$$

(Localization)

PRIME SPECTRUM

$\vdash Z(0)$

$Z(x), Z(y) \vdash Z(x+y)$

Additive subgroup

$Z(x) \vdash Z(xy)$

$Z(y) \vdash Z(xy)$

Multiplicative closure (ideal)

$Z(1) \vdash$

$Z(xy) \vdash Z(x), Z(y)$

Prime

Distributive lattice
dual to
that for the Zariski
spectrum

CONSTRUCTIBLE SPECTRUM

Axioms for

D and/or Z

plus

$Z(x), D(x) \vdash \ldots$

$\vdash Z(x), D(x)$

Boolean algebra generated by either distributive lattice

CHARACTERIZATION

$$D(a_1) \ldots D(a_n) \vdash D(b_1) \ldots D(b_n)$$

if and only if

we have

$$a_1^{r_1} \ldots a_m^{r_m} = \sum \lambda_j b_j$$

that is

$$\mathrm{Mult}(a_i) \between \mathrm{Idl}(b_j)$$

(Similarly in other cases.)

As spaces

$$\begin{tikzcd}
 & \mathrm{Const}(A) \arrow[dl] \arrow[dr] & \\
\mathrm{Spec}(A) & & \mathrm{Prime}(A)
\end{tikzcd}$$

surjections

Issue

$$X \longrightarrow X'$$

spaces with the same points and different topologies

Typical functional analysis situation.

Development in ASD?

CASE STUDY

Suppose $f : A \hookrightarrow B$ is an integral extension. Then the going up theorem holds for f

Going up theorem

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[rr] & & \operatorname{Spec} A \\
Q' \arrow[rr, dashed, mapsto] & & P' \\
\cup\!| & & \cup\!| \\
Q \arrow[rr, mapsto] & & P
\end{tikzcd}$$

Integral extension

For all $b \in B$ we have some

$$b^n + a_{n-1} b^{n-1} + \cdots + a_0 = 0$$

over A

IDEAL INTERPRETATION

of going up in terms of formal spaces.

$$
\begin{tikzcd}
(\mathrm{Spec}\,B \leq \mathrm{Spec}\,B) \arrow[rr] \arrow[dr] \arrow[dd] & & (\mathrm{Spec}\,A \leq \mathrm{Spec}\,A) \arrow[dd, "\mathrm{Snd}\ \left(\substack{\text{order}\\ \text{reversal}}\right)"] \\
& \bullet \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & \\
\mathrm{Spec}\,B \arrow[rr] & & \mathrm{Spec}\,A
\end{tikzcd}
$$

where

$$
\begin{tikzcd}
Y \leq_X Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\leq"] & Z \arrow[d] \\
Y \arrow[r] & X
\end{tikzcd}
$$

universal.

(Cf. Krull dimension à la Joyal)

MAIN CONTENT

If $f: A \hookrightarrow B$ is an integral extension

then $\operatorname{Spec} f: \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is

surjective.

In formal space terms

suppose $a_1 \ldots a_n, c_1 \ldots c_m \in A$

then $D(a_1) \ldots D(a_n) \vdash D(c_1) \ldots D(c_m)$

in $\operatorname{Spec} B$

entails

$D(a_1) \ldots D(a_n) \vdash D(c_1) \ldots D(c_m)$

in $\operatorname{Spec} A$.

IDEAL ARGUMENT

$$
\begin{tikzcd}
\mathrm{Prime}(B/q) \arrow[rr, dashed] \arrow[dr, "!"] \arrow[dd, dashed, no head] & & \mathrm{Prime}(A/p) \arrow[dd, dotted] \\
& \mathrm{Prime}((\ )^* A/p) \arrow[ur, dotted, no head] \arrow[dl, dotted, no head] & \\
\mathrm{Prime}B \arrow[rr, "(\ )"'] & & \mathrm{Prime}\, A
\end{tikzcd}
$$

$B/q \longrightarrow (\ )^* A/p$

integral

A/p

(for as usual order)

PROOF OBJECTS

A partially ordered (or preordered) set already has a categorical aspect.

$$p : x \leq y \quad q : y \leq z \vdash q \circ p : x \leq z .$$

A category is like a bicategory or more ... .

$$\begin{array}{ccccc} \mathbb{C}(b,c) & \times & \mathbb{C}(a,b) & \longrightarrow & \mathbb{C}(a,c) \\ g & , & f & \longmapsto & g \circ f \end{array}$$

plus a proof object

$$\alpha : h \circ (g \circ f) \xrightarrow{\cong} (h \circ g) \circ f$$

plus more.

? Algebraic version of a quasi-category. (Joyal)

VALUATION RING
(OF A FIELD) Coquand-Persson

K a field. For $a \in K^\times$:

$\vdash x \cdot y$ (when $xy = x+y$)

$\vdash x, x^{-1}$

$x \vdash -x$

$x, y \vdash xy$

Characterization

$a_1 \ldots a_n \vdash b_1 \ldots b_m$

if and only if

we have a $\mathbb{Z}$-polynomial equation

$$f(\underline{a}, \underline{b}) = b_1^{d_1} \cdots b_m^{d_m} + f_1(\underline{a}, \underline{b}) = 0$$

lower terms in $\underline{b}$

ELIMINATION THEORY

Correctness of the characterization depends on

$$\frac{A \overset{f}{\vdash} B, c \qquad c, D \overset{g}{\vdash} E}{A, D \vdash B, E}$$

The resultant $\mathrm{Res}_c(f, g)$ with respect to c of the polynomials f, g.

RESULTANTS

Example

$$f(x) = a_0 + a_1 x + a_2 x^2$$

$$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

$$\mathrm{Res}_x(f, g)$$

$$= \begin{vmatrix} a_0 & a_1 & a_2 & & \\ & a_0 & a_1 & a_2 & \\ & & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & \\ & b_0 & b_1 & b_2 & b_3 \end{vmatrix}$$

Common place

$\mathrm{Res}_x(f, g) = 0$ iff "$\exists \alpha \; f(\alpha) = g(\alpha) = 0$"

but in fact .... .

ASSOCIATIVITY

Take $f(x)$, $h(x,y)$, $g(y)$

Then

$$Res_y(Res_x(f,h),g) = Res_x(f, Res_y(h,g))$$

Semi-proof

$Res_y(Res_x(f,h),g) = 0$ iff

$\exists \beta.\ Res_x(f(x), h(x,\beta)) = 0 \wedge g(\beta) = 0$

iff

$\exists \beta\, \exists \alpha\ \ f(\alpha) = 0 \wedge h(\alpha,\beta) = 0 \wedge g(\beta) = 0$

iff

. . . . . $Res_x(f, Res_y(h,g)) = 0$

ZARISKI SPECTRUM

(Joyal)

$\vdash D(1)$ $\qquad D(ab) \vdash D(a)$ $\qquad D(0) \vdash$

$D(a), D(b) \vdash D(ab)$ $\qquad D(ab) \vdash D(b)$ $\qquad D(a+b) \vdash D(a), D(b)$

Characterization

$$D(a_1) \dots D(a_n) \vdash D(b_1) \dots D(b_m)$$

if and only if

$$\mathrm{Mult}(a_i) \between \mathrm{Idl}(b_j)$$

ELEMENTARY ALGEBRA

Correctness of characterization depends on

$$\frac{A \vdash X, d \qquad d, B \vdash Y}{A, B \vdash X, Y}$$

$a = x + \lambda d$

$d^n b = y$

$$a^n b = (a^{n-1} + a^{n-2}(\lambda d) + \cdots + (\lambda d)^{n-1})\, b x + \lambda^n y$$

From the trivial

If $\quad a = b + c$

then $\quad a^n = \frac{a^n - c^n}{a - c} \cdot a - c \quad + c^n$

$$= (a^{n-1} + \cdots + c^{n-1})\, b \; + c^n.$$

? ASSOCIATIVITY ?

No!

Instead there is a relation between the expressions obtained by eliminating in different orders.

A shadow of higher identity types?