

FORMAL TOPOLOGY

AND

COMMUTATIVE

ALGEBRA

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3WF Top

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CONSTRUCTIVE MATHEMATICS

Two influences

Martin-Löf Type theory as a
compelling, elegant setting.

Bishop Constructive mathematics
looks natural / normal.

What should constructive

Commutative Algebra

Algebraic Geometry

look like?

TOPOS / LOCALE

THEORY ANALYSIS

(Joyal, Mulvey)

Constructive variants
of classical results in
terms of locales.

(Classifying toposes.)

CONSTRUCTIVE CONTENT

of classical commutative algebra
(& much else)

(Loquand, Lombardi et al)

Extract the algebraic manipulations
etc hidden in classical arguments
using ideal objects, Zorn's Lemma etc.

Hilbert's Programme

Ideal elements as eliminable
in proofs of concrete statements.

CONSTRUCTIVE COMMUTATIVE ALGEBRA

Should be conducted in a predicative constructive setting
(Mathi-Löf)

Ideal elements should appear so it looks normal (Bishop!)

IDEA Use some form of formal topology. Ideal elements as the 'points' of a formal space.
But then how to argue?

Analogy with ASD

OLD TECHNIQUE

(Joyal, Moerdijk, Wraith)

Extend a topos

(adding ideal elements,
making it Boolean....);

prove something in the
extended (simpler)
situation;

derive information about
the original topos (descent).

DREAM

Explain such techniques
via 'the' logic of all forcing
extensions.

Example: Cardinal collapse

One can extend to make any
set countable and (Toyal)
via an open projection.

So anything might as well be
countable.

(So the Dream logic is weak:
certainly predicative.)

WHAT IS A
PRIME IDEAL?

Constructively
an 'element' of some
'formal space'.

BUT
WHICH?

ZARISKI SPECTRUM

(Joyal)

$$\vdash D(1)$$

Multiplicative
set

$$D(x), D(y) \vdash D(xy)$$

$$D(xy) \vdash D(x)$$

Substitution

$$D(xy) \vdash D(y)$$

$$D(0) \vdash$$

Prime

$$D(x+y) \vdash D(x), D(y)$$

(Localization)

PRIME SPECTRUM

$$\vdash \mathbb{Z}(0)$$

Additive
subgroup

$$\mathbb{Z}(x), \mathbb{Z}(y) \vdash \mathbb{Z}(x+y)$$

$$\mathbb{Z}(x) \vdash \mathbb{Z}(xy)$$

Multiplicative
closure
(ideal)

$$\mathbb{Z}(y) \vdash \mathbb{Z}(xy)$$

$$\mathbb{Z}(1) \vdash$$

Prime

$$\mathbb{Z}(xy) \vdash \mathbb{Z}(x), \mathbb{Z}(y)$$

Distributive lattice

dual to

that for the Zariski
spectrum

CONSTRUCTIBLE SPECTRUM

Axioms for

D and/or Z

plus

$Z(x), D(x) \vdash \dots$

$\vdash Z(x), D(x)$

Boolean algebra
generated
by either distributive
lattice

CHARACTERIZATION

$$D(a_1) \dots D(a_n) \vdash D(b_1) \dots D(b_n)$$

if and only if
we have

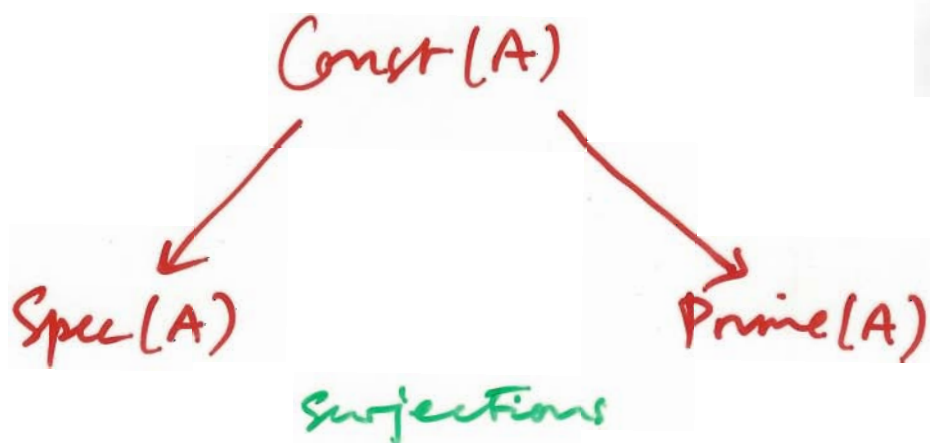
$$a_1^{r_1} \dots a_n^{r_n} = \sum \lambda_j b_j$$

that is

$$\text{Mult}(a_i) \preceq \text{Idl}(b_j)$$

(Similarly in other cases.)

As spaces



ISSUE

$$X \longrightarrow X'$$

spaces with the same
points and different
topologies

Typical functional analysis
situation.

Development in ASD?

CASE STUDY

Suppose $f: A \hookrightarrow B$ is an integral extension. Then the going up theorem holds for f

Going up theorem

$$\begin{array}{ccc} \text{Spec } B & \longrightarrow & \text{Spec } A \\ \mathfrak{q}' & \dashrightarrow & \mathfrak{p}' \\ \vdots & & \vdots \\ \mathfrak{q} & \longrightarrow & \mathfrak{p} \end{array}$$

Integral extension

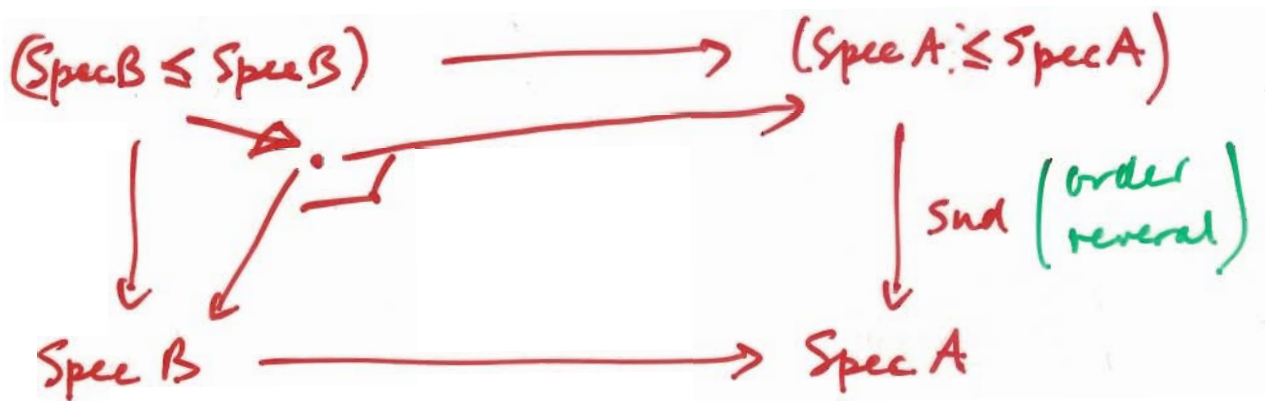
For all $b \in B$ we have some

$$b^n + a_{n-1}b^{n-1} + \dots + a_0 = 0$$

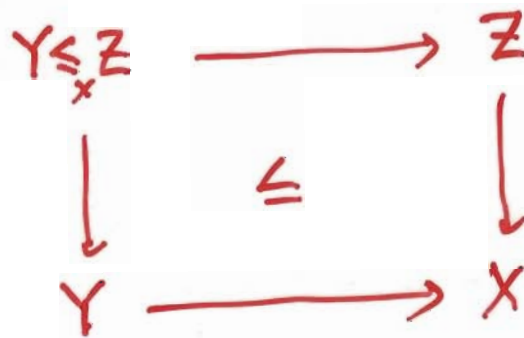
over A

IDEAL INTERPRETATION

of going up in terms of formal spaces.



where



universal.

(Cf. kohl dimension à la Joyal)

MAIN CONTENT

If $f: A \hookrightarrow B$ is an integral extension

then $\text{Spec} f: \text{Spec} B \rightarrow \text{Spec} A$ is

surjective.

In formal space terms

suppose $a_1, \dots, a_n, c_1, \dots, c_m \in A$

then $D(a_1) \dots D(a_n) + D(c_1) \dots D(c_m)$

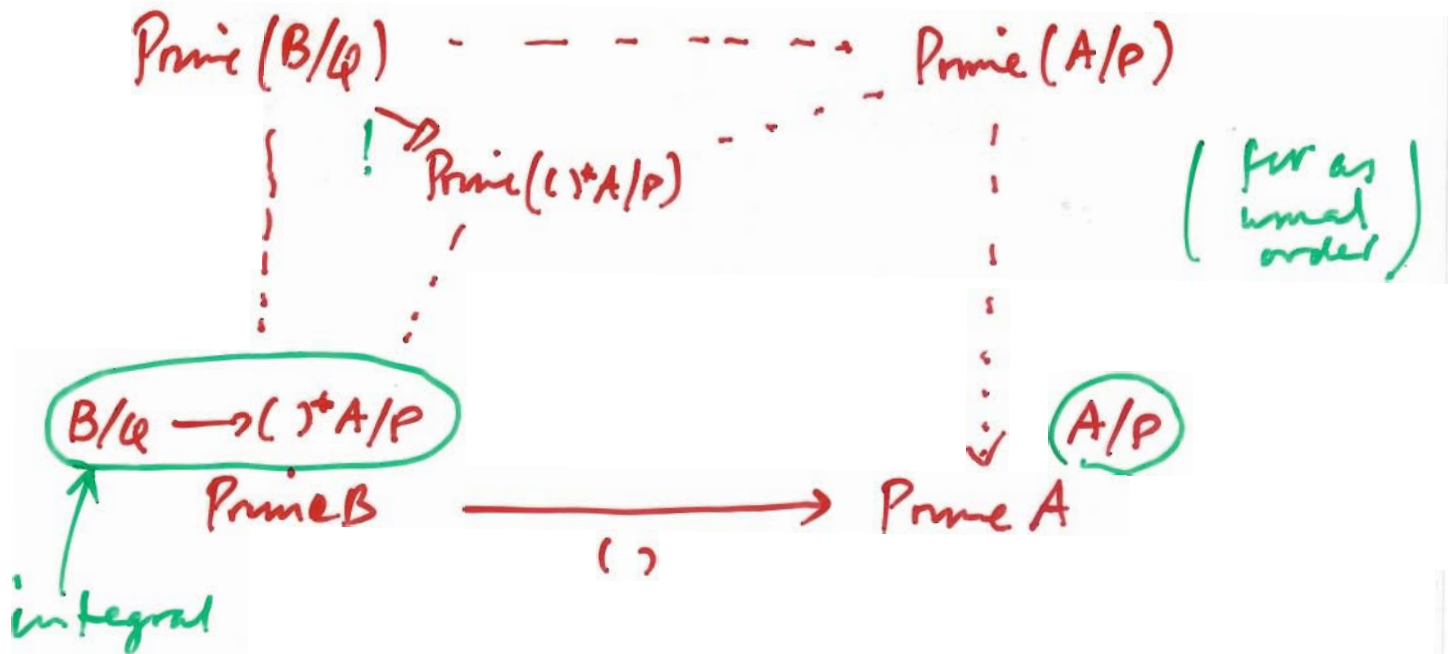
in $\text{Spec} B$

entails

$D(a_1) \dots D(a_n) + D(c_1) \dots D(c_m)$

in $\text{Spec} A$.

IDEAL ARGUMENT



PROOF OBJECTS

A partially ordered (or preordered) set already has a categorical aspect.

$$p: x \leq y \quad q: y \leq z \quad \vdash \quad q \circ p: x \leq z.$$

A category is like a bicategory or more ...

$$\begin{array}{ccc} \mathcal{C}(b, c) \times \mathcal{C}(a, b) & \longrightarrow & \mathcal{C}(a, c) \\ g, f & \longmapsto & g \circ f \end{array}$$

plus a proof object

$$\alpha: \eta_0(g \circ f) \xrightarrow{\sim} (\eta_0 g) \circ f$$

plus more.

? Algebraic version of a quasi-category. (Joyal)

VALUATION RING

(OF A FIELD) *Coquand - Persson*

K a field. For $a \in K^\times$:

$$\vdash x \cdot y \quad (\text{when } xy = x+y)$$

$$\vdash x, x^{-1}$$

$$x \vdash -x$$

$$x, y \vdash xy$$

Characterization

$$a_1 \dots a_n \vdash b_1 \dots b_m$$

if and only if


we have a \mathbb{Z} -polynomial equation

$$f(\underline{a}, \underline{b}) = b_1^{d_1} \dots b_m^{d_m} + f_1(\underline{a}, \underline{b}) = 0$$

↑
lower terms in \underline{b}

ELIMINATION THEORY

Correctness of the characterization depends on

$$\begin{array}{r} f \\ A \vdash B, c \quad c, D \vdash E \\ \hline A, D \vdash B, E \end{array}$$


The resultant $\text{Res}_c(f, g)$ with respect to c of the polynomials f, g .

RESULTANTS

Example

$$f(x) = a_0 + a_1 x + a_2 x^2$$

$$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

$$\text{Res}_x(f, g)$$

$$= \begin{vmatrix} a_0 & a_1 & a_2 & & \\ & a_0 & a_1 & a_2 & \\ & & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & \\ & b_0 & b_1 & b_2 & b_3 \end{vmatrix}$$

Common place

$$\text{Res}_x(f, g) = 0 \text{ iff } \exists \alpha \text{ } f(\alpha) = g(\alpha) = 0$$

but in fact

ASSOCIATIVITY

Take $f(x)$, $h(x,y)$, $g(y)$

Then

$$\text{Res}_y (\text{Res}_x (f, h), g) = \text{Res}_x (f, \text{Res}_y (h, g))$$

Semi-proof

$$\text{Res}_y (\text{Res}_x (f, h), g) = 0 \quad \checkmark$$

$$\exists \beta. \text{Res}_x (f(x), h(x, \beta)) = 0 \wedge g(\beta) = 0 \quad \checkmark$$

$$\exists \beta \exists \alpha \quad f(\alpha) = 0 \wedge h(\alpha, \beta) = 0 \wedge g(\beta) = 0 \quad \checkmark$$

$$\dots \text{Res}_x (f, \text{Res}_y (h, g)) = 0$$

ZARISKI SPECTRUM

(Joyal)

$$\vdash D(1)$$

$$D(ab) \vdash D(a)$$

$$D(0) \vdash$$

$$D(a), D(b) \vdash D(ab)$$

$$D(ab) \vdash D(b)$$

$$D(a+b) \vdash D(a), D(b)$$

Characterization

$$D(a_1) \dots D(a_n) \vdash D(b_1) \dots D(b_m)$$

if and only if

$$\text{Mult}(a_i) \leq \text{Mult}(b_j)$$

ELEMENTARY ALGEBRA

Correctness of characterization depends on

$$\frac{A \vdash X, d \quad d, B \vdash Y}{A, B \vdash X, Y}$$

$a = x + \lambda d$ $d^n b = y$

$$a^n b = (a^{n-1} + a^{n-2}(\lambda d) + \dots + (\lambda d)^{n-1}) b x + \lambda^n y$$

From the trivial

$$\text{If } a = b + c$$

$$\begin{aligned} \text{then } a^n &= \frac{a^n - c^n}{a - c} \cdot a - c + c^n \\ &= (a^{n-1} + \dots + c^{n-1}) b + c^n. \end{aligned}$$

? ASSOCIATIVITY ?

No!

Instead there is a relation between the expressions obtained by eliminating in different orders.

A shadow of higher identity types?