SEQUENTIALITY

Robin Cockett

Martin Hyland

Andrea Schalk

1. Causal Interaction.
2. Sequential Interaction.
3. Towards Games.

ABSOLUTE CAUSALITY SPACES

DEFINITION

An (absolute) causality space is

$$\langle\ |\ \rangle : U \times X \longrightarrow R$$

such that

$$\langle u|x\rangle = \langle u'|x'\rangle$$

implies

$$\begin{array}{ccc} \langle u|x'\rangle & = & \langle u'|x'\rangle \\ \parallel & & \\ \langle u|x\rangle & = & \langle u'|x\rangle \end{array}$$

EXAMPLES

Games

$$U \times X \longrightarrow R$$

Players strategies

Opponents strategies

Resulting play

Abstract Games (some cases)

$$U \times X \longrightarrow R$$

Player's functions

Opponent's functions

Complete course of computation

CAUSAL MAPS

DEFINITION

A causal map is

$$\begin{tikzcd} U \arrow[d, "f"'] & \times & X \arrow[r] & R \\ V & \times & Y \arrow[u, "F"'] \arrow[r] & S \end{tikzcd}$$

such that

$$\langle u \mid F(y) \rangle = \langle u' \mid F(y) \rangle \implies \langle f(u) \mid y \rangle = \langle f(u') \mid y \rangle$$

$$\langle f(u) \mid y \rangle = \langle f(u) \mid y' \rangle \implies \langle u \mid F(y) \rangle = \langle u \mid F(y') \rangle.$$

EXAMPLE

Maps of games as causal maps

A map $\sigma : A \to B$ of games is a strategy in $A^\perp \parr B$.

Then $f : \underset{\text{P in A}}{U} \longrightarrow \underset{\text{P in B}}{V}$ is just composition with σ:

$$u : I \to A \longmapsto \sigma \circ u : I \to B$$

and $F : Y \longrightarrow X$ is similar.

Intuition for causal condition!

THEOREM

The category of absolute causality spaces is $*$-autonomous

(In fact models full linear logic.)

INTERNAL FUNCTION SPACE

If $A = (U \times X \to R)$

$B = (V \times Y \to S)$

then $A \multimap B =$

$$\left(\{(f,F) \mid \text{causal maps}\} \times (U \times Y) \longrightarrow R \times S \right)$$

$$\langle (f,F) \mid (u,y) \rangle = \left(\langle u \mid F(y) \rangle , \langle f(u) \mid y \rangle \right).$$

FACT

Over Sets, absolute causality spaces have a full *-autonomous subcategory equivalent to 'weak totality spaces'.

(so not much gain save the conceptual; however quite different uses of the internal logic, so … .)

RELATIVE CAUSALITY SPACES

DEFINITION

A relative causality space is

$$\langle\ |\ \rangle : U \times X \longrightarrow R$$

with $\leq$ on U, $\leq$ on X

such that

for $u \leq u'$, $x \leq x'$

$$\langle u|x\rangle = \langle u'|x'\rangle \Rightarrow \langle u|x\rangle = \langle u|x'\rangle = \langle u'|x'\rangle, \quad \langle u|x\rangle = \langle u'|x\rangle = \langle u'|x'\rangle$$

MORE RESTRICTED VERSION

$\leq$ on U, $\leq$ on X, $\leq$ on R

and $\langle\ |\ \rangle$ order preserving.

RELATIVELY CAUSAL MAPS

DEFINITION

A relatively causal map is

$$\begin{tikzcd} U \times X \arrow[r] \arrow[d, "f"'] & R \\ V \times Y \arrow[r] \arrow[u, "F"'] & S \end{tikzcd}$$

such that

for $u \le u'$, $y \le y'$

$$\langle u \mid F(y) \rangle = \langle u' \mid F(y) \rangle \implies \langle f(u) \mid y \rangle = \langle f(u') \mid y \rangle$$

$$\langle f(u) \mid y \rangle = \langle f(u) \mid y' \rangle \implies \langle u \mid F(y) \rangle = \langle u \mid F(y') \rangle.$$

EXAMPLES

Games with weaker notion of result
(eg. the set of moves independent of order)

Coherence spaces

$$U \times X \longrightarrow R$$

cliques in A   cocliques in A $|A|_{\perp}$

$$(u, x) \longmapsto u \cap x$$

THEOREM

The category of relative causality spaces is *-autonomous. (In fact it models full linear logic.)

SEQUENTIALITY

GENERAL APPROACH Take some causality category + add a sequential condition.

DEFINITION

A sequentiality space is a causality space

$$\langle\ |\ \rangle : U \times X \to R$$

$$\leq_m U \ , \ \leq_m X$$

such that

$$\boxed{\text{for } u \leq u' \quad x \leq x' \qquad \langle u|x\rangle = \langle u|x'\rangle \ \text{ or } \ \langle u|x\rangle = \langle u'|x\rangle .}$$

EXPLANATION

GAME INTERPRETATION

Suppose $\langle u|x\rangle$ a finite play.

If Player just played then however much further Player is prepared to go, the game will progress no further without input from Opponent.

So $\langle u|x\rangle = \langle u'|x\rangle$ all $u \leq u'$

Similarly if Opponent just played then $\langle u|x\rangle = \langle u|x'\rangle$ all $x \leq x'$.

SEQUENTIAL MAPS

DEFINITION

A sequential map

$$\begin{tikzcd} U \times X \arrow[r] \arrow[d, "f"'] & R \\ V \times Y \arrow[u, "F"'] \arrow[r] & S \end{tikzcd}$$

is an (order-preserving) map of causality spaces such that

for all $u \leq u'$ $\quad y \leq y'$

$$\langle u \mid F(y) \rangle = \langle u' \mid F(y) \rangle \text{ or } \langle f(u) \mid y \rangle = \langle f(u) \mid y' \rangle .$$

EXPLANATION

GAME INTERPRETATION

We play the P-strategy (f, F) in $A \multimap B$ against the O-strategy (u, y). Suppose the resulting pairs

$$\big(\langle u \mid F(y)\rangle, \langle f(u) \mid y\rangle\big)$$

are finite. If we stopped in $A^{\perp}$; then further effort by O in B will make no difference, so

$$\langle f(u) \mid y\rangle = \langle f(u) \mid y'\rangle \text{ all } y \leq y'.$$

Similarly if we stopped in B then

$$\langle u \mid F(y)\rangle = \langle u' \mid F(y)\rangle \text{ all } u \leq u'.$$

SEQUENTIAL ORDER ON MAPS

DEFINITION

Suppose (f,F) and (g,G) are maps

$$(U \times X \to R) \longrightarrow (V \times Y \to S).$$

Set $(f,F) \leq (g,G)$ if & only if $\big[(f,F) \leq (g,G)$ pointwise & $\big]$

$$\boxed{\begin{array}{c} \text{for all } u \leq u' \quad y \leq y' \\[1em] \left\{\begin{array}{c} \langle u \mid F(y) \rangle = \langle u' \mid F(y') \rangle \\ \& \\ \langle f(u) \mid y \rangle = \langle f(u') \mid y' \rangle \end{array}\right. \\[1em] \text{or} \\[1em] \left\{\begin{array}{c} \langle u \mid F(y) \rangle = \langle u' \mid G(y) \rangle \\ \& \\ \langle f(u) \mid y \rangle = \langle g(u') \mid y \rangle \end{array}\right. \\[1em] \text{or} \\[1em] \left\{\begin{array}{c} \langle u \mid F(y) \rangle = \langle u \mid G(y') \rangle \\ \& \\ \langle f(u) \mid y \rangle = \langle g(u) \mid y' \rangle \end{array}\right. \end{array}}$$

EXPLANATION

GAME INTERPRETATION

1st case O has just played so doesnt matter how much further O goes

2nd case P has just played in B.

3rd case P has just played in $A^\perp$.

Observation:

In games this order coincides with the point-wise order.

THEOREM

The category of sequentiality spaces is $*$-autonomous.

(In fact it models full linear logic.)

GLIMPSE OF PROOF

Show eg

$$\frac{A \otimes B \longrightarrow C^{\perp}}{A \longrightarrow (B \multimap C^{\perp})}$$

by showing both equivalent to symmetric $A \otimes B \otimes C \longrightarrow \perp$.

Symmetric conditions.

for all $u \leq u'$ $v \leq v'$ $w \leq w'$

$$\langle u | f(v,w) \rangle = \langle u' | f(v,w) \rangle \ \& \ \langle v | g(u,w) \rangle = \langle v' | g(u,w) \rangle$$

OR

$$\langle u | f(v,w) \rangle = \langle u' | f(v,w) \rangle \ \& \ \langle w | h(u,v) \rangle = \langle w' | h(u,v) \rangle$$

OR

$$\langle v | g(u,w) \rangle = \langle v' | g(u,w) \rangle \ \& \ \langle w | h(u,v) \rangle = \langle w' | h(u,v) \rangle.$$

EXAMPLES

QUESTION Which of these embed in the subcategories of sequential spaces and maps?

1. Usual categories of games

   SEQUENTIAL ✓

2. Typical categories of abstract games

   NOT SEQUENTIAL ✓

EXAMPLES

CONTINUED

(3) Coherence spaces

Hypercoherence spaces

SEQUENTIAL!

(4) Games and non-deterministic strategies

NOT SEQUENTIAL?

Misleading: can be fixed with different embedding.

(Non-determinism vs Concurrency)

TOWARDS GAMES

ISSUES

- Detecting individual moves
- Sequential order v. pointwise order
- Opponent strategies in $A \multimap B$

‘RESTRICTION’

Suppose play $\langle u | x \rangle$

Then there should be minimal

$$\bar{u} = u|_x \leq u$$

$$\bar{x} = x|_u \leq x$$

such that

$$\langle \bar{u} | \bar{x} \rangle = \langle u | x \rangle .$$

Axioms in these terms

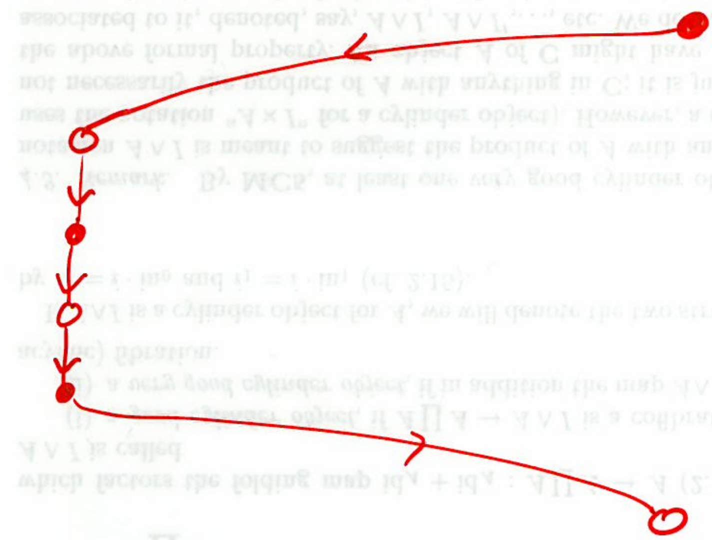
- detect individual moves
- make the sequential order pointwise.

OLD PROBLEM

(REVERSED)

When playing

$A^{\perp} ⅋ B$



O cannot change sides, but eg. when about to play in B can use information about what happened in $A^{\perp}$. SO MUCH MORE THAN $U \times Y$

COMPLETING (DOMAINS OF) STRATEGIES

ONE SOLUTION (works in examples)

Use absolute sequentiality.

Complete under sequential $\leq$ and $\bigvee^{\uparrow}$ (preserving $\perp$).

(Not quite as stated: the orthogonality $\perp$ plays a bigger role.)

WHY CAN THIS WORK?

(4. Totality Spaces)

ONWARD

Relation with other work in H/Ong/Wallen project.

Nickau/Ong investigate large #s of flavours of game.

Hope to detect these via abstractly defined subcategories.