ABSTRACT AND CONCRETE MODELS OF RECURSION

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$\boxed{\text{The abstract is concrete.}}$

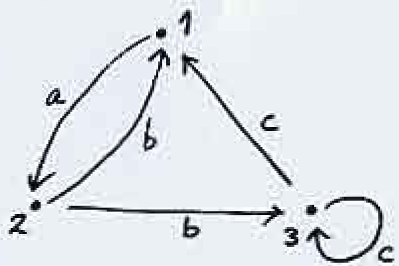
PLAN

1. Background, aims
2. Category theory
3. Feedback, fixed points
4. Feedback, matrices.

FINITE AUTOMATA

Traversing a diagram

EXAMPLE



Suppose 1 is initial state
and 3 is terminal state

Then
$$a\,(ba)^*\,b\,(c\,(ab)^*)^*$$
is accepted.

RESULT OF CALCULATION

(WARNING. Traditional matrix convention gives words in reverse.)

$$\begin{pmatrix} 0 & b & c \\ a & 0 & 0 \\ 0 & b & c \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} (c^*ba)^* & (c^*ba)^*c^*b & (c^*ba)^*cc^* \\ (ac^*b)^*a & (ac^*b)^* & (ac^*b)^*acc^* \\ ((ba)^*c)^*b(ab)^*a & ((ba)^*c)^*b(ab)^* & ((ba)^*c)^* \end{pmatrix}$$

What is this?

REGULAR LANGUAGES

Σ a finite alphabet

Σ^* set of words in Σ

A language is a subset of Σ^*.

The regular languages are those generated from singleton letters x, y, z

by

empty set	0
union	$+$
singleton trivial word	1
concatenation	$\cdot$
star	$(\ )^*$

(where $a^* = 1 + a + a^2 + \cdots$)

KLEENE'S THEOREM

Fix a finite alphabet Σ. The languages recognized by a finite automaton over Σ are exactly the regular languages.

WHY ???

WHY IS IT INTUITIVELY OBVIOUS ?

THEOREM

which encapsulates the characterization direction

There is a traced monoidal functor

$$\underline{\mathrm{Aut}} \longrightarrow \underline{\mathrm{Reg}}$$

category based on automata

category of matrices of regular languages

AIM To explain this!

ALGEBRA OF REGULAR OPERATIONS

$$0+a = a = 0+a$$
$$a+(b+c) = (a+b)+c$$
$$a+b = b+a$$

$$1.a = a = 1.a$$
$$a.(b.c) = (a.b).c$$

$$0.a = 0 = a.0$$
$$(a+b).c = a.c+b.c \qquad a.(b+c) = a.b+a.c$$

$$(a.b)^* = 1 + a(b.a)^* b$$

$$(a+b)^* = (a^* b)^* a^*$$

$$a^{**} = a^*$$

$$(a^n)^*(1+a+\cdots+a^{n-1}) = a^*$$

Conway Algebras

$$0+a = a = 0+a$$
$$a+(b+c) = (a+b)+c$$
$$a+b = b+a$$

$$1.a = a = 1.a$$
$$a.(b.c) = (a.b).c$$

$$0.a = 0 = a.0$$
$$(a+b).c = a.c+b.c \qquad a.(b+c) = a.b+a.c$$
$$(a.b)^* = 1 + a(b.a)^* b$$
$$(a+b)^* = (a^* b)^* a^*$$

CLAIM This should be the fundamental concept. WHY?

Exercises

1. What are the small Conway Algebras?

2. Show that

$$a^{**} = a^*$$
and
$$(a^n)^*(1+a+\cdots+a^{n-1}) = a^*$$

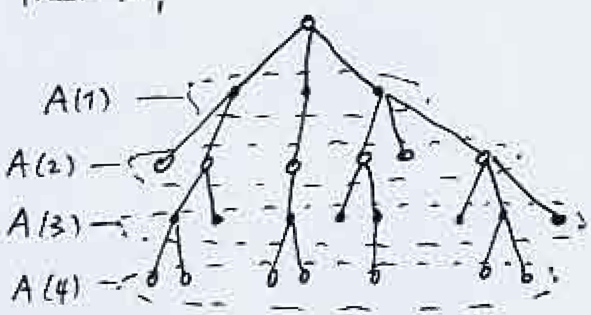
do not follow from the axioms for Conway Algebras.

GAMES

as trees or forests

$A(1) \leftarrow A(2) \leftarrow A(3) \cdots$

That is,

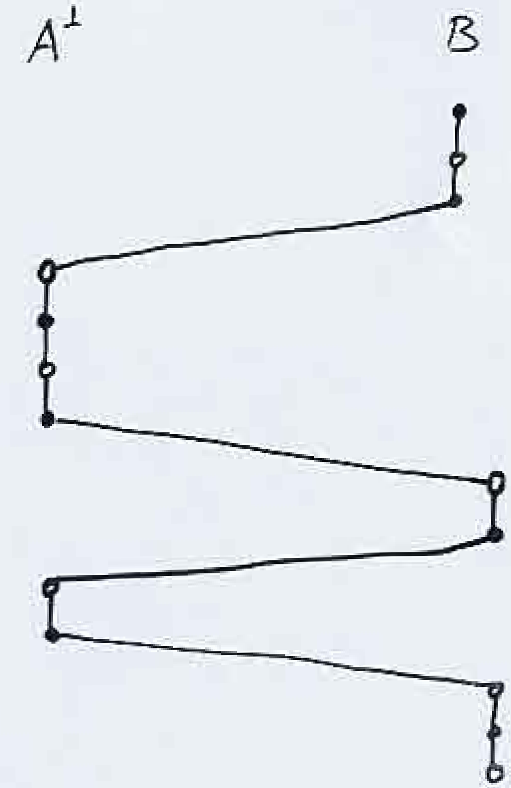


Here the opponent starts and player / opponent alternate.

So Opponent plays the odd
Player plays the even stages.

LINEAR FUNCTION SPACE

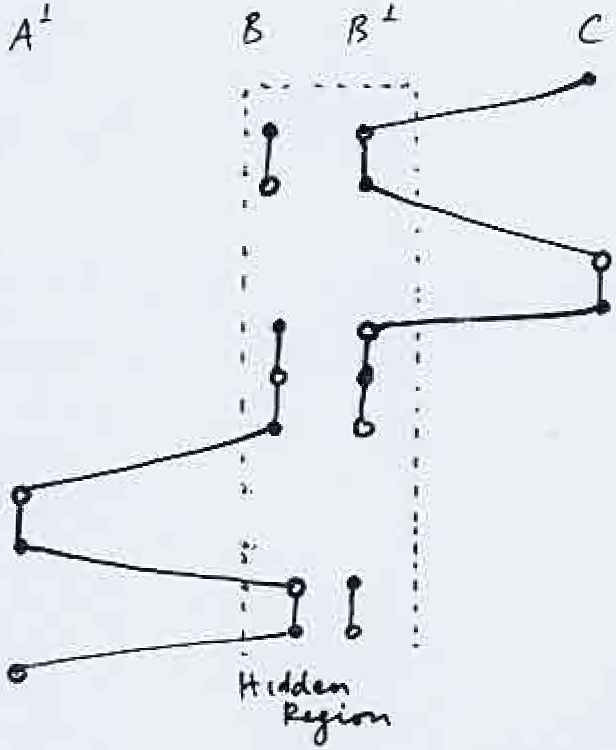
$$A \multimap B$$



Composition of Strategies

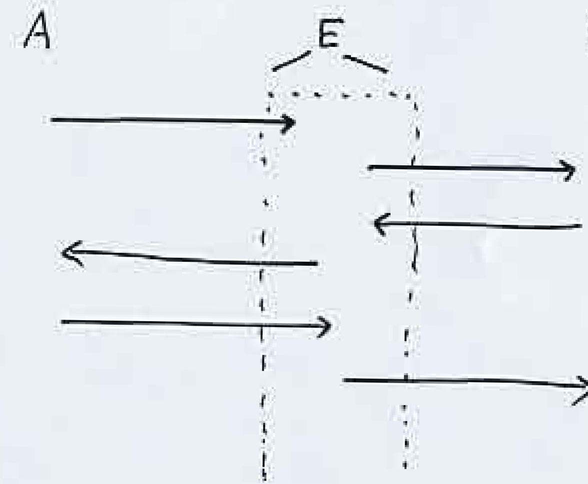
(Player Strategies)

$$\frac{A \overset{\sigma}{\multimap} B \qquad B \overset{\tau}{\multimap} C}{A \overset{\sigma;\tau}{\multimap} C}$$



Security

The shape of an attack



The attack
hidden from
A and B

Pavlovic : Category of cord processes

Cf papers of Durgin, Mitchell, Pavlovic

History-Free Strategies

A_- opponent tokens in A

A_+ player tokens in A

Strategy is partial function

$$A_- \longrightarrow A_+$$

with good properties

So strategy in $A \multimap B$

is partial function

$$A_+ + B_- \longrightarrow A_- + B_+$$

History-Free Composition

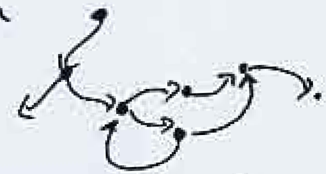
$$\begin{tikzcd}
A_+ \arrow[r, bend right] \arrow[d, bend left] \arrow[dr, phantom, "\sigma"] & A_- \\
B_+ \arrow[r, bend right] \arrow[d, bend left] \arrow[dr, phantom, "\tau"] & B_- \arrow[u, bend left] \arrow[l, bend right] \\
C_+ & C_- \arrow[u, bend left] \arrow[l, bend right]
\end{tikzcd}$$

$$A_+ + B_- \xrightarrow{\sigma} A_- + B_+$$

$$B_+ + C_- \xrightarrow{\tau} B_- + C_+$$

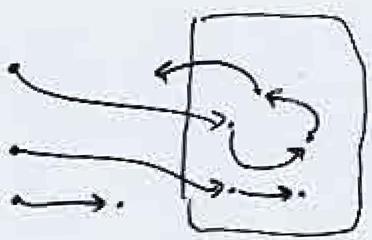
CONTRASTS

Automaton



follow all runs of a token through a diagram

Composition



each point goes to a visible point, iterating within the hidden region until it becomes visible

RANGE OF MODELS

Automata

Flow diagrams

Circuits

Interactive systems

Action structures

Proofs

Diagrammatic methods pervade computer science. Is there a unifying point of view?