INTERACTION OF STRATEGIES

BY

COMPOSITION OF INTERLEAVINGS

$$\frac{A \xrightarrow{\sigma} B \xrightarrow{\tau} C}{A \xrightarrow[\sigma;\tau]{} C}$$

$$\frac{\Gamma \vdash^{\sigma} \Delta, A \qquad A, \Pi \vdash^{\tau} \Sigma}{\Gamma, \Pi \vdash^{(\sigma;\tau)_A} \Delta, \Sigma}$$

Interaction by plugging

$$\boxed{\sigma} - \boxed{\tau} - \boxed{\rho}$$

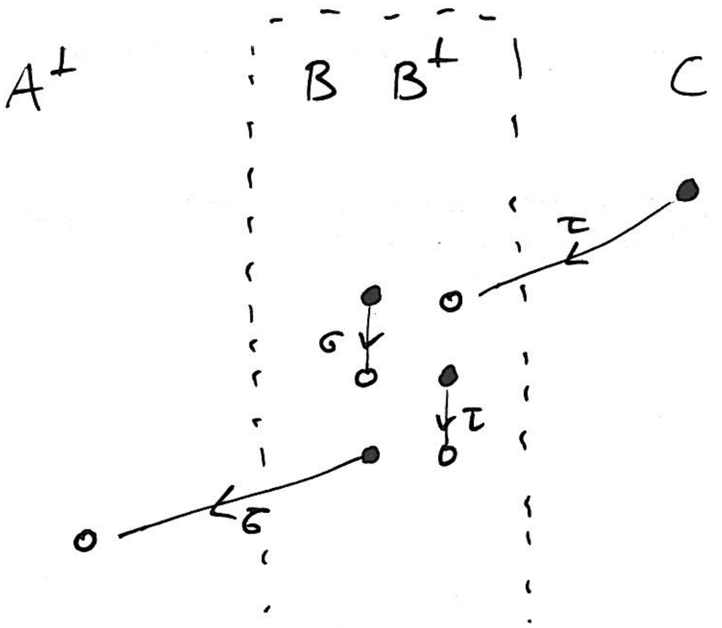
clearly associative

BUT

PARALLEL COMPOSITION PLUS HIDING

$$A \xrightarrow{\sigma} B \xrightarrow{\tau} C$$

defined via



intuitively associative.

HOWEVER

(1) It is much less clear for some restricted kinds of strategies

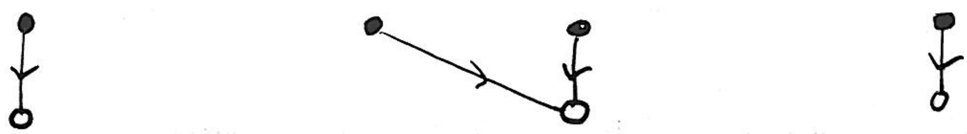
(2) There is the Blass Problem.

BLASS EXAMPLE : SEQUENTIAL

$A \xrightarrow{\sigma} B \qquad B \xrightarrow{\tau} C \qquad C \xrightarrow{\rho} D$

Initial move

Response



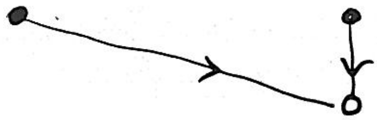
①

$A \xrightarrow{\sigma ; \tau} C$

$A \xrightarrow{(\sigma ; \tau) ; \rho} D$

②

$B \xrightarrow{\tau ; \rho} D$



$A \xrightarrow{\sigma ; (\tau ; \rho)} D$

COMPOSITION VIA PLAYS
(STANDARD VERSION)

A strategy $\sigma : A$ is identified with the plays (= sequences of moves/positions) in A to which it gives rise.

So $\sigma : A \multimap B$ is a sequence from A, B

Consider sequences w from A, B, C. Then

$$\sigma ; \tau = \{ (w)_{A,C} : (w)_{A,B} \in \sigma \text{ and } (w)_{B,C} \in \tau \} .$$

For associativity consider sequences from A, B, C, D : one has to reconstruct these from various restrictions.

COMPOSITION VIA PLAYS
(REVISED VERSION)

A sequence u from A, B is an interleaving of a sequence r from A and s from B:

$$r \xrightarrow{u} s$$

<u>CLAIM</u> There is a category whose maps are (appropriate) interleavings.

Then we have

$$\sigma ; \tau = \{ r \xrightarrow{u} s \xrightarrow{v} t \mid u \in \sigma \ v \in \tau \},$$

and this obviously associative.

MERGING : GENERALITIES

Consider preordered sets P, Q : a merge is given by embeddings $\begin{matrix} P \searrow \\ \quad P+Q \\ Q \nearrow \end{matrix}$ into a preorder on the disjoint sum.

Equivalently by relations

$$(q \leq p) \qquad P \times Q^{op} \longrightarrow 2 \qquad P \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\,} Q$$

$$(p \leq q) \qquad Q \times P^{op} \longrightarrow 2 \qquad Q \underset{r'}{\longrightarrow\!\!\!\!\!\!\!+\;\,} P$$

such that

$$r ; r' \vdash id_P \ (= \leq_P) \qquad p \leq q \leq p' \vdash p \leq p'$$

$$r' ; r \vdash id_Q \ (= \leq_Q) \qquad q \leq p \leq q' \vdash q \leq q'.$$

(Then the merge on $P+Q$ is a 2-colimit.)

Conditions

Asynchrony P, Q posets and $\quad p \leq q \leq p' \vdash p < p'$
$\qquad\qquad q \leq p \leq q' \vdash q < q'$

Sequentiality P, Q linearly ordered and
$\qquad p \leq q$ if + only if not $q \leq p$.

COMPOSITION OF MERGES

$$P \underset{r'}{\overset{r}{\rightleftarrows}} Q \underset{s'}{\overset{s}{\rightleftarrows}} R$$

gives

$$P \underset{s';r'}{\overset{r;s}{\rightleftarrows}} R$$

where we have e.g.

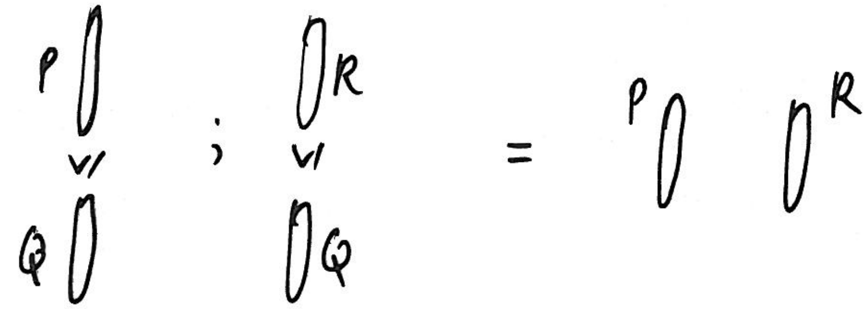
$$p \le q \le r \le q' \le p' \Rightarrow p \le q \le q' \le p' \Rightarrow p \le q \le p' \Rightarrow p \le p'$$

So there is a category of preordered sets and merges.

WARNING.

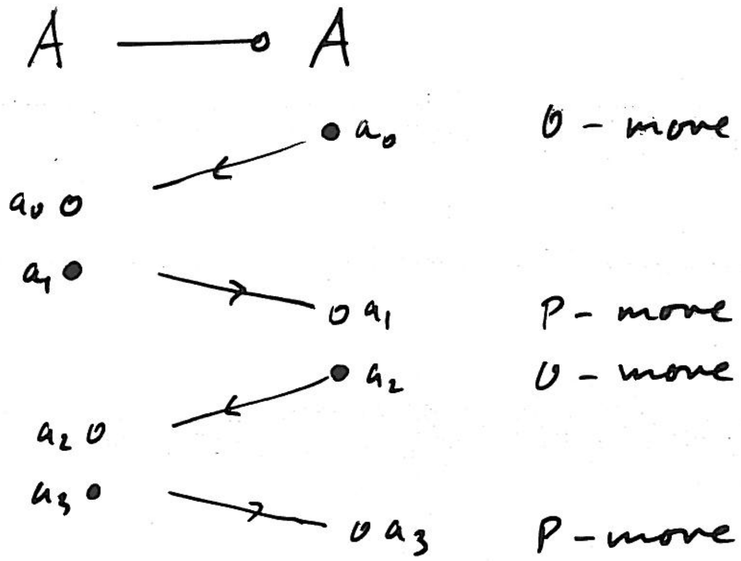
Asynchrony is preserved

Sequentiality is not



MOTIVATION : COPY-CAT

In games the identity is a copycat strategy



i.e. we interleave in a twisted sense one way for Opponent moves the other for Proponent moves.

Simple merging has an identity which synchronises the moves.

INTERLEAVING OF PLAYS

A play in a game A is already a merge $P = (P_1 \leftrightarrows P_0)$ of Opponent moves P_1 and Proponent moves P_0.

Interleaving is given by Proponent data only:

$$\begin{tikzcd} P_1 \arrow[r, shift left] \arrow[d] & P_0 \arrow[l, shift left] \\ Q_1 \arrow[r, shift left] & Q_0 \arrow[l, shift left] \arrow[u] \end{tikzcd}$$

such that

$\boxed{\wedge}$ $\quad q_1 \leq p_1 \leq p_0 \leq q_0 \vdash q_1 \leq q_0$

$\boxed{\vee}$ $\quad p_0 \leq q_0 \leq q_1 \leq p_1 \vdash p_0 \leq p_1$

(This induces merges

$$P_1 \rightleftarrows Q_1$$

$$P_0 \rightleftarrows Q_0$$

$$P_1 \rightleftarrows Q_0$$

$$Q \rightleftarrows P_0$$

but that is derived structure.)

COMPOSITION OF INTERLEAVINGS

$$\begin{tikzcd}
P_1 \arrow[r, leftrightarrow, "+" description] \arrow[d, "+" description] & P_0 \\
Q_1 \arrow[r, leftrightarrow, "+" description] \arrow[d, "+" description] & Q_0 \arrow[u, "+" description] \\
R_1 \arrow[r, leftrightarrow, "+" description] & R_0 \arrow[u, "+" description]
\end{tikzcd}$$

composes in the obvious way as e.g.

$$r_1 \leq q_1 \leq p_1 \leq p_0 \leq q_0 \leq r_0 \vdash r_1 \leq q_1 \leq q_0 \leq r_0 \vdash r_1 \leq r_0$$

In the asynchronous (sequential) case the identity is now COPY-CAT.

SEQUENTIAL GAMES

Take plays $(P_1 \leftrightarrow P_0)$ which are (finite)

- asynchronous sequential
- Opponent starting $\exists p_1 \, \forall p_0 \; p_1 \leq p_0$
- alternating $p_1 < p_1' \vdash \exists p_0 \; p_1 < p_0 < p_1'$

  $p_0 < p_0' \vdash \exists p_1 \; p_0 < p_1 \leq p_0'$

Take interleavings

$$\begin{array}{ccc} P_1 & \leftrightarrow & P_0 \\ \downarrow & & \uparrow \\ Q_1 & \leftrightarrow & Q_0 \end{array}$$

such that in the $\multimap$ game

- Opponent starts $\exists q_1 \, \forall p_1 \; q_1 \leq p_1$
- Proponent chooses where to play

$$\forall p_1 \, q_0 . \quad \begin{array}{ll} \underline{\text{either}} & \exists p_0 \; p_1 \leq p_0 \leq q_0 \\ \underline{\text{or}} & \exists q_1 \; q_0 \leq q_1 \leq p_1 \end{array}$$

This is the <u>sequential setting</u>.

OBSERVATIONS IN SEQUENTIAL SETTING

① The order of Opponent moves now also forced i.e.

$$\forall q_1\, p_0 \quad \underline{\text{either}} \quad \exists p_1 \quad q_1 \leq p_1 \leq p_0$$
$$\underline{\text{or}} \quad \exists q_0 \quad p_0 \leq q_0 \leq q_1$$

(cf. the switching convention is forced)

Moreover the induced merges totally order $P_1 + P_0 + Q_1 + Q_0$.

② The interleaving properties are closed under composition.

So we get an alternative view on associativity in the standard cases.

ABSTRACT GAMES

Models of linear logic constructed thus

$$\begin{array}{ccc} \text{Geometry of interaction} & \text{eg.} & \text{Traced mon-cats.} \\ \Downarrow \text{Int} & & \\ \text{Degenerate models} & \text{eg.} & \text{Compact closed cats} \\ \Downarrow \text{Glueing} & & \\ \text{"Non-interactive" models} & & \\ \Downarrow \text{Orthogonality} & & \\ \text{"Interactive" models} & & \end{array}$$

often "read" as categories whose objects are a kind of game.

Examples

Non-interactive : GRel, 'Concurrent games' Abramsky Melliès

. . . . .

Interactive : Coherence spaces, Hypercoherences

. . . . .

? GHCoh ?

CONCRETE NON-SEQUENTIAL GAMES

(SIMPLE VERSION : SKETCH)

A game A is a collection of plays $(P_1 \longleftrightarrow P_0)$ closed under initial segment (& ...).

A strategy $\sigma : A$ is a collection of plays again closed under initial segment and <u>consistent for Proponent</u> i.e.

suppose p_0 in $(P_1 \longleftrightarrow P_0) \in \sigma$

p_0' in $(P_1' \longleftrightarrow P_0') \in \sigma$

with $\{p_1 : p_1 \leq p_0\} = \{p_1' : p_1' \leq p_0'\}$

((as preordered sets))

then there is play $(P_1'' \longleftrightarrow P_0'') \in \sigma$

$$\underbrace{\text{with } \quad p_0, p_0' \in \quad \& \quad \{p_1'' : p_1'' \leq p_0\} = \{p_1'' : p_1'' \leq p_0'\}}$$

[in <u>exact</u> asynchronous case this is $p_0 = p_0'$]

(Idea: Petri-net style computation)

MULTIPLICATIVE STRUCTURE
(SIMPLEST CASE)

Plays in $B \multimap C$ are given by all interleavings of plays $(Q_1 \leftrightarrow Q_0)$ in B with plays $(R_1 \leftrightarrow R_0)$ in C.

Then plays in $A \multimap (B \multimap C)$ correspond exactly to plays in $A \otimes B \multimap C$: more symmetrically, plays in $A \parr B \parr C$ are interleavings of plays in A, B, C via data satisfying

six conditions of form

$$p_1 \leq q_0 \leq q_1 \leq p_0 \vdash p_1 \leq p_0$$

and six conditions of form

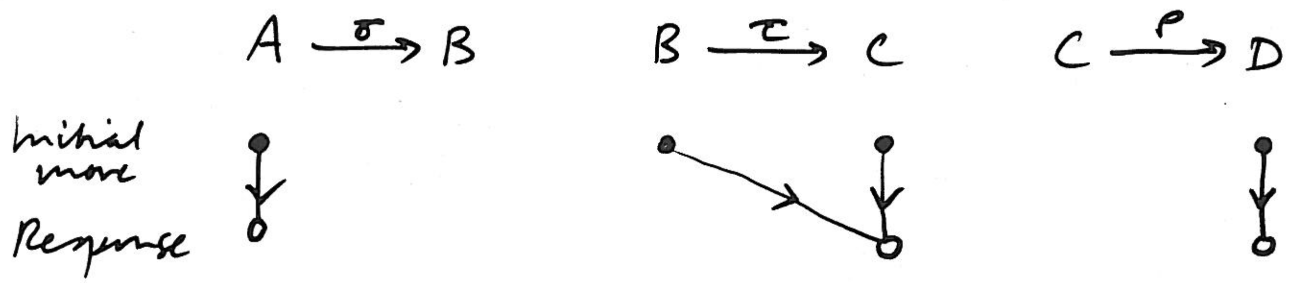
$$p_1 \leq r_0 \leq r_1 \leq q_0 \vdash p_1 \leq q_0$$

Take maps $A \to B$ to be strategies $\sigma : A \multimap B$ and we have a

$*$-autonomous category

(in fact model for full linear logic)

BLASS EXAMPLE : NON-SEQUENTIAL



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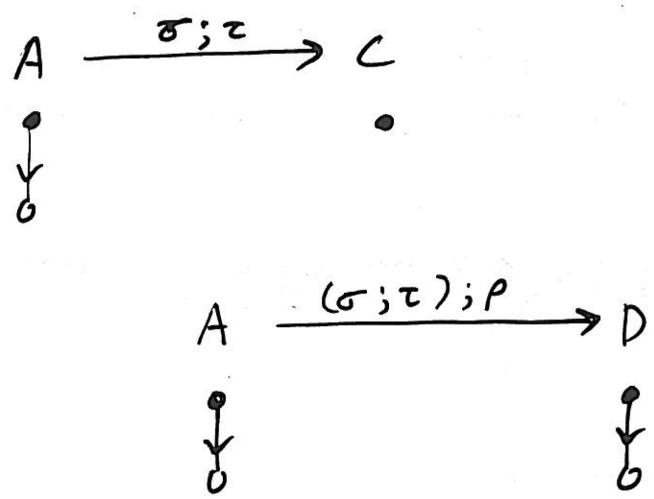

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