GAMES AND LOGIC

Descriptive Set Theory

Model Theory

Verification

Do there exist strategies such that ?

Semantics

Structure of strategies

$\boxed{\text{Composition}}$

CONNECTIONS

The point of the Ecole de Printemps

BUT why is it even plausible?

- Semantics ~

  abstract approach to combinatorial complexity

- Example! Luke Ong.

PLAN

Basic game semantics

- composition of strategies
- intuitionistic linear logic
- abstract interpretation
- Kleisli category

Appendices

- Conway games
- abstract non-determinism

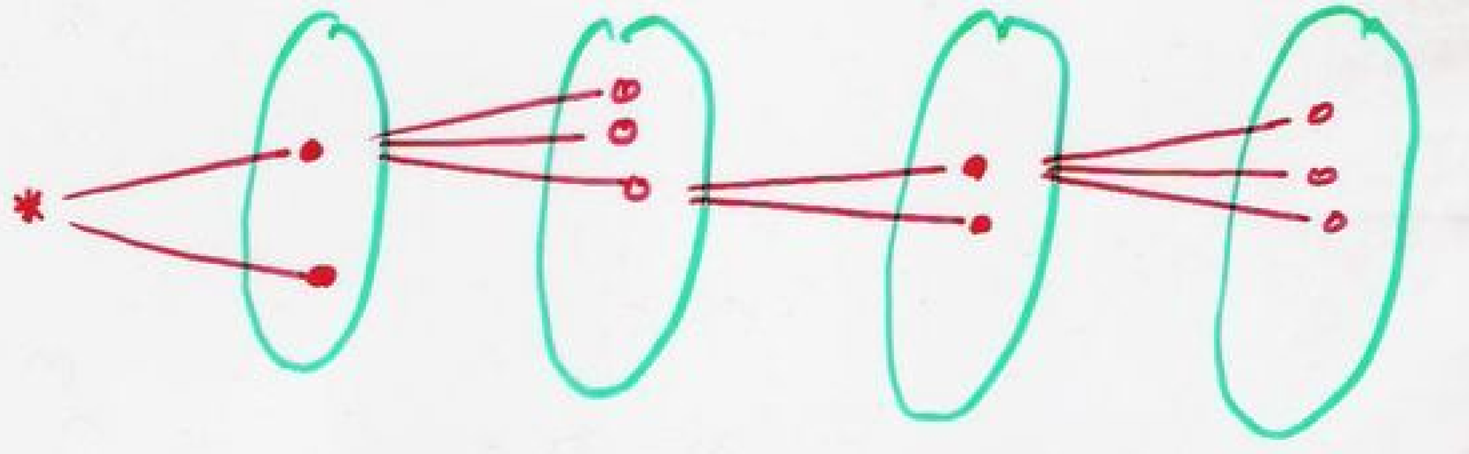
Pointer games

- general theory
- views and innocent strategies
- composition
- full abstraction

TREES

Diagrams

$$1 = T(0) \leftarrow T(1) \leftarrow T(2) \leftarrow T(3) \leftarrow T(4) \cdots$$



So $T \in \mathrm{Trees} \cong [\mathbb{N}^{op}, \mathrm{Set}]$

with $\mathbb{N} = \{1 \le 2 \le 3 \le \cdots\}$.

There is an initial position $*$, then

T(1) the first moves which can be made

T(2) the next moves + so on

(I identify moves with the position reached as a result.)

Write $a \to b$ for $\overset{\text{move}}{\text{position}}$ b follows a

(opposite of the predecessor relation)

SIMPLE GAMES & STRATEGIES

A game is $A \in \mathbf{Trees}$ with understanding

$A(\text{odd})$ moves played by Opponent O
$A(\text{even})$ moves played by Proponent P

Proponent strategies σ (partial, simple non-deterministic) given by

- relation from O-positions to next P-positions s.t.
- a subtree $\hat{\sigma} \subseteq A$ of possible plays s.t.
- a set of P-positions s.t.

Take $e : \mathbb{N} \longrightarrow \mathbb{N}$ embedding of evens.
Then $\sigma \subseteq e^*(A) \in \mathbf{Trees}$

EXERCISES 1

1.1 $s : \mathbb{N} \to \mathbb{N}$ induces $s^* : \text{Trees} \to \text{Trees}$

$$(T(1) \leftarrow T(2) \cdots) \longmapsto (T(2) \leftarrow T(3) \cdots)$$

with left adjoint $s_!$

$$(T(1) \leftarrow T(2) \cdots) \longmapsto (T(1) \leftarrow T(1) \leftarrow T(2) \cdots)$$

and right adjoint s_*

$$(T(1) \leftarrow T(2) \cdots) \longmapsto (1 \leftarrow T(1) \leftarrow T(2) \cdots)$$

For A a game describe $s_! A$ and $s_* A$ as games in terms of A.

1.2 Take a strategy $\sigma \subseteq e^* A$. Show that the resulting $\hat{\sigma} \subseteq A$ lies in a pullback

$$\begin{tikzcd}
e_* \sigma \arrow[r] & e_* e^* A \\
\hat{\sigma} \arrow[u] \arrow[r] & A \arrow[u, "\eta"']
\end{tikzcd}$$

in Trees.

Show also that the concrete relation is

$$a \xrightarrow{\sigma} b \text{ iff only if } a \to b \in \hat{\sigma}(\text{even}) = \sigma(\cdot).$$

1.3 Take $o : \mathbb{N} \to \mathbb{N}$ embedding of odds.

$\mathbb{N} \underset{o}{\overset{e}{\rightrightarrows}} \mathbb{N}$ induces $e^* A \longrightarrow o^* A$. Show

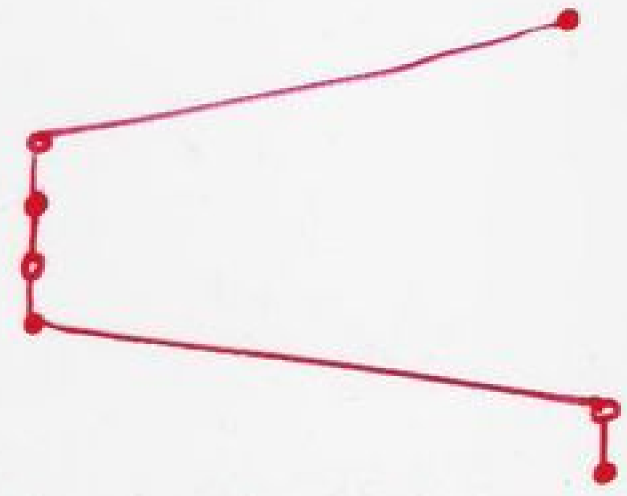
σ deterministic iff $\sigma \to e^* A \to o^* A$ monic.

LINEAR FUNCTION SPACE

(playing $A^{\perp}$ and B in parallel)

$$A \multimap B$$

typical play



swap discipline 0111100

A finite sequence ε of 0s and 1s

is $\multimap$ if + only if $\varepsilon(1) = 0 \quad \varepsilon(2r+1) = \varepsilon(2r)$

$|\varepsilon|_0$ number of zeros

$|\varepsilon|_1$ number of ones

$|\varepsilon| = |\varepsilon|_0 + |\varepsilon|_1$ length

$$A \multimap B(n) = \sum_{\substack{|\varepsilon| = n \\ \varepsilon \multimap}} A(|\varepsilon|_1) \times B(|\varepsilon|_0)$$

with predecessor arising from maps

$$A(r+1) \times B(s) \longrightarrow A(r) \times B(s)$$

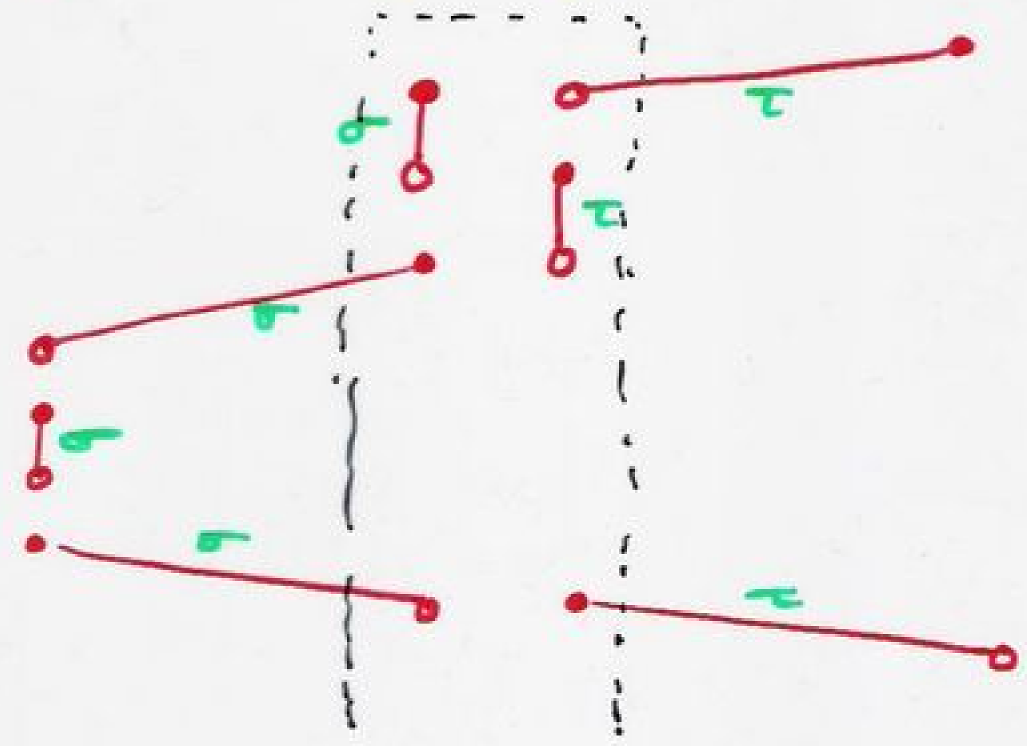
$$A(r) \times B(s+1) \longrightarrow A(r) \times B(s)$$

<u>Definition</u> A linear map $\sigma : A \longrightarrow B$ is a strategy in $A \multimap B$.

COMPOSITION OF STRATEGIES

(by parallel composition plus hiding)

$A \xrightarrow{\sigma} B \quad B \xrightarrow{\tau} C$



γ sequence of		$A \multimap B$	$B \multimap C$	$A \multimap C$
2s	A	1		1
1s	B	0	1	
0s	C		0	0

determines γ_{AB} γ_{BC} γ_{AC} sequences of 0s, 1s.

γ is composition if and only if $\gamma(1) = 0$

$$\gamma(2n) = 0 \Rightarrow \gamma(2n+1) = 0$$
$$\gamma(2n+1) = 2 \Rightarrow \gamma(2n+2) = 2$$
$$\gamma(2n) = 1 \Rightarrow \gamma(2n+1) = 1 \text{ or } 2$$
$$\gamma(2n+1) = 1 \Rightarrow \gamma(2n+2) = 0 \text{ or } 1.$$

COMPOSITION OF STRATEGIES CTD.

The composition 'sequences' are at length n

$$\sum_{|\gamma|=n} A(|\gamma|_2) \times B(|\gamma|_1) \times C(|\gamma|_0)$$

Then $(\gamma; a, b, c)$ is in accord with σ, τ if and only if

$$(\gamma_{AB}, a, b) \in \hat{\sigma}$$

$$(\gamma_{BC}, b, c) \in \hat{\tau}$$

and then if $|\gamma_{AC}|$ even we have

$$(\gamma_{AC}, a, c) \in \sigma;\tau$$

This defines the composite $\sigma;\tau$

EXERCISES 2

2.1 The copy-cat sequences are
$\iota = (01100110 \ldots\ldots)$ and
we take $\mathrm{id}(n) = \{(\iota, a, a) : a \in A(n)\}$

$$\subseteq A \multimap A\,(2n)$$

Show that id is a two-sided identity for composition.

2.2 Show that id is a deterministic strategy.

2.3 Show that composition of strategies is associative.

2.4 Show that the composite of two deterministic strategies is deterministic.

NB There is an alternative approach via a theory of interleavings.

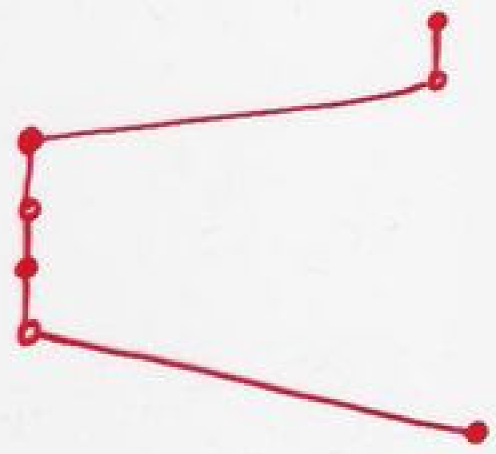
INTUITIONISTIC LINEAR LOGIC

Multiplicative structure $\multimap$ and

Tensor product (play games in //)

$$A \quad \otimes \quad B$$

typical play



$$A\otimes B(n) = \sum_{\substack{|\varepsilon| = n \\ \varepsilon\ \otimes}} A(|\varepsilon|_1) \times B(|\varepsilon|_0)$$

Additive structure

Product (O chooses a game & we play it)

$$A \quad \overset{\&}{\times} \quad B$$

EITHER  OR 

$$A \times B(0) = 1$$

$$A \times B(n) = A(n) + B(n)$$

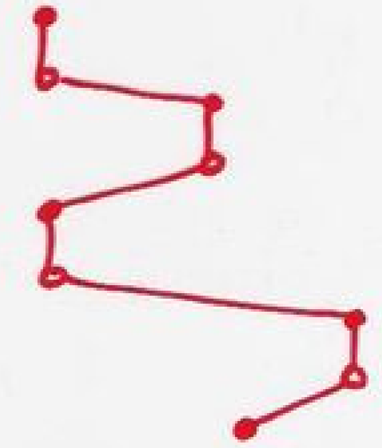
INTUITIONISTIC LINEAR LOGIC CTD

Exponential structure

Simple version of $!A$ play in as many copies of A as the Opponent chooses.

$\dagger A$: $A \quad A \quad A \quad \ldots$

typical play



A sequence α of 0s, 1s, 2s, ... is $\dagger$ if + only if $\alpha(r) = k+1 \implies \exists\, s<r \;\; \alpha(s) = k$ and $\alpha(2r+2) = \alpha(2r+1)$.

Let m_α be the maximal value of α so

$$|\alpha| = |\alpha|_0 + \cdots\cdots + |\alpha|_{m_\alpha}$$

Then

$$!A(n) = \sum_{\substack{|\alpha| = n \\ \alpha \dagger}} \prod_{0 \le i \le m_\alpha} A(|\alpha|_i)$$

EXERCISES 3

3.1 What does it mean for a sequence ε to be ∞?

3.2 Show that the empty game

$$I(0) = 1 \qquad I(n+1) = 0$$

is the unit for $\otimes$.

3.3 Show that there is a bijection

$$\frac{A \otimes B \longrightarrow C}{A \longrightarrow (B \multimap C)}$$

natural in A.

3.4 Show that $\times$ is a categorical product. What is the terminal?

3.5 Show how to equip $\dagger$ with the structure

$$\varepsilon : \dagger A \longrightarrow A$$

$$\delta : \dagger A \longrightarrow \dagger\dagger A$$

$$e : \dagger A \longrightarrow I$$

$$d : \dagger A \longrightarrow \dagger A \otimes \dagger A$$

of a linear exponential comonad.

(You need to look this up!)

THE RELATIONAL MODEL
(OF LINEAR LOGIC)

The category **Rel** of sets and relations has

- a tensor product $A \otimes B = A \times B$
  & linear function space $A \multimap B = A \times B$
- products $A \& B = A + B$
- a linear exponential comonad $!A = M(A)$
  (the collection of finite multisets in A)

making it a model of (classical) linear logic.

RELATIONAL INTERPRETATION

Given a game A write

$$|A| = \sum_{n=0}^{\infty} A(n)$$

for the collection of positions.

For $\sigma : A \longrightarrow B$ given by $\sigma(n)$ let $|\sigma| : |A| \longrightarrow\!\!\!\!\!\!+\;\; |B|$ be the relation defined by

$a \, |\sigma| \, b$ if + only if there is ε with

$$(\varepsilon, a, b) \in \sigma(n) \subseteq A \multimap B\,(2n).$$

Then

- $|\;|$ is a functor Games $\longrightarrow$ Rel
- $|\;|$ is monoidal and linearly distributive

Furthermore if we restrict to deterministic strategies

- $|\;|$: Games $\longrightarrow$ Rel is faithful.

(See HS for an abstract interpretation which is full + faithful.)

THE KLEISLI CATEGORY

Objects : $A, B \ldots$ (usual games)

Maps $A \to B$: $\sigma : {!A} \multimap B$

As
$$\frac{!A \longrightarrow B \times C}{!A \longrightarrow B \qquad !A \longrightarrow C}$$

$B \times C$ is still the product.

! property $\boxed{!(A \times B) \cong {!A} \otimes {!B}}$

As
$$\frac{\dfrac{!(A \times B) \longrightarrow C}{!A \otimes !B \longrightarrow C}}{!A \longrightarrow (!B \multimap C)}$$

$!B \multimap C$ is a function space $B \Rightarrow C$

Cartesian closed category

~ Typed λ-calculus

ABSTRACT HARMER-McCUSKER NON-DETERMINISM

Objects $A = (A_0\ ,\ A_1\ ,\ A_\infty)$ sets of

P-pos^ns O-poss ∞-plays

$$(A \multimap B)_0 = (A_0 \times B_0 + A_1 \times B_1)$$

$$(A \multimap B)_1 = A_0 \times B_1$$

$$(A \multimap B)_\infty = (A_\infty \times B_1 + A_0 \times B_\infty + A_\infty \times B_\infty)$$

Maps $\sigma : A \longrightarrow B$

$\sigma_0 \subseteq (A \multimap B)_0$ $\quad \sigma_1 \subseteq (A \multimap B)_1$ $\quad \sigma_\infty \subseteq (A \multimap B)_\infty$

reachable player position $\quad$ position of non term^n $\quad$ genuine ∞ plays

This gives a model for linear logic!

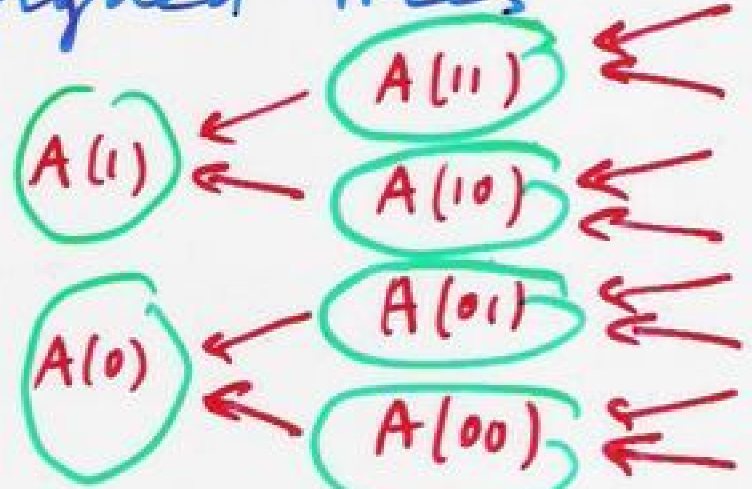
(Critical is $A_0 \multimap B_\infty$ $\ B_\infty \multimap C_1$)

There is a good abstraction functor to this from Harmer - McCusker.

CONWAY GAMES

Let Bin be the category (poset) of binary trees under extension

$[\mathrm{Bin}^{op}, \mathrm{Set}]$ is STree the category of signed trees



Obvious duality $A \longleftrightarrow A^*$ $(0 \leftrightarrow 1)$

and

$$A \otimes B(\varepsilon) = \sum_{M(\varepsilon_1, \varepsilon_2; \varepsilon)} A(\varepsilon_1) \times B(\varepsilon_2)$$

Play in same way but no swap discipline is enforced.

(Previous theory derives from this.)

POINTER GAMES

For a game A the game $\#A$ is given by sequences $\underline{a} = (a_1, \ldots a_n)$ of moves from A together with pointers ϕ pointing to an earlier predecessor move. (Backtracking.)

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad \ldots$$

Also have $\#^{\bullet}A$ the single threaded version with only one opening move.

- $\#$ and $\#^{\bullet}$ extend to maps and give functors.
- $\#A \cong \dagger\,\#^{\bullet}A$

A Conway version of the simple exponential.

POINTERS

Strictly decreasing maps

$$\phi : \{1, \ldots, n\} \longrightarrow \{0, \ldots, n-1\}$$

And

$$\# A(n) = \{ (\underline{a}, \phi) \mid a_{\phi(i)} \to a_i \text{ in } A \}$$

Pointer order

$\phi < \psi$ if and only if

each $\phi(i) \in \{ \psi^k(i) \}$.

(So predecessor π is maximal in the order.)

Interpretation of $\#\# A$ (ETC)

$$\boxed{\#\# A(n) = \{ (\underline{a}, \phi_1, \phi_2) \mid \phi_1 \geq \phi_2 \text{ and } (\underline{a}, \phi_2) \in \# A \}}$$

ETC

HARMER'S SYMMETRIC MONOIDAL CLOSED CATEGORY

Additive implication $A \mathbin{-\!\boxplus} B$

$$A \mathbin{-\!\boxplus} B(1) = B(1) = A(0) \times B(1)$$

$n > 1$ $$A \mathbin{-\!\boxplus} B(n+1) = A(n) \times B(1) + B(n+1)$$

Category

Objects $A, B, \ldots$ (usual games)

Maps $\sigma : \#(A \mathbin{-\!\boxplus} B)$

(Composition has an abstract explanation as

$$\#(A \mathbin{-\!\boxplus} B) \cong \dagger(\dagger\#' A \multimap \#' B)$$
)

CARTESIAN CLOSED CATEGORY

Objects $A, B \ldots$ (usual games)

Maps $\sigma : \#^{\cdot}(A \mathrel{-\!\!\boxplus} B)$

Composition comes from

$$\#^{\cdot}(A \mathrel{-\!\!\boxplus} B) \cong \#A \multimap \#^{\cdot}B$$
$$\cong \dagger\#^{\cdot}A \multimap \#^{\cdot}B$$

So we have the Kleisli category for $\dagger$ on objects of form $\#^{\cdot}A$

Then

$$\#^{\cdot}(A \times B) \cong \#^{\cdot}A \times \#^{\cdot}B$$

shows that $A \times B$ a product

and

$$\#^{\cdot}(B \mathrel{-\!\!\boxplus} C) \cong \dagger\#^{\cdot}B \multimap C$$

shows that $B \mathrel{-\!\!\boxplus} C$ gives a function space.

VIEWS

Let $\underline{a} = (\underline{a}, \phi)$ be a position in #A : the P view $\ulcorner \underline{a} \urcorner$ is given by :

n odd : $\phi(n) = m$

$$\ulcorner(a_1 \dots a_n)\urcorner = \ulcorner(a_1 \dots a_m)\urcorner \, a_n$$

n even : $\phi(n) = m$

$$\ulcorner(a_1 \dots a_n)\urcorner = \ulcorner(a_1 \dots a_{n-1})\urcorner \, a_n$$

pointing to residual of a_m if it exists

So we make as if all O-moves are immediate successors.

Plays satisfy visibility just when all pointers required exist.

IDEA OF INNOCENCE

Play according to the view.

COMPUTATIONAL LINEAR LOGIC

$!A$ consists of $(\underline{a}, \phi) \in \# A$

with $\phi(2r+2) = 2r+1$

(only O backtracks)

$?A$ consists of $(\underline{a}, \phi) \in \# A$

with $\phi(2r+1) = 2r$

(only P backtracks)

- $!$ is a linear exponential comonad
- $?$ is a linear exponential monad for $!$

$$\boxed{?(A \multimap\!\boxplus B) \cong\, !A \multimap ?B}$$

- There is a distributive law

$$\boxed{!?A \longrightarrow ?!A}$$

- $?$ preserves products

$$\boxed{?(A \times B) \cong ?A \times ?B}$$

INNOCENT STRATEGIES

(The data for) An innocent strategy in A is

$$\sigma : ?(A)$$

(this gives the response to news)

Innocent strategies operate in general pointer games via a natural transformation

$$\dagger ?A \longrightarrow \#^{\bullet} A$$

(with appropriate properties)

Composition

$$\frac{!A \xrightarrow{\sigma} ?B \qquad !B \xrightarrow{\tau} ?C}{!A \to !!A \xrightarrow{!\sigma} !?B \longrightarrow ?!B \xrightarrow{?\tau} ??C \longrightarrow ?C}$$

uses the distributive law.

CARTESIAN CLOSED STRUCTURE

Products

$$
\begin{array}{c}
!A \longrightarrow ?(B \times C) \\
\hline
!A \longrightarrow ?B \times ?C \\
\hline
!A \longrightarrow ?B \qquad !A \longrightarrow ?C
\end{array}
$$

Function space

$$
\begin{array}{c}
!(A \times B) \longrightarrow ?C \\
\hline
!A \otimes !B \longrightarrow ?C \\
\hline
!A \longrightarrow (!B \multimap ?C) \\
\hline
!A \longrightarrow ?(B \mathrel{-\!\!\boxplus} C)
\end{array}
$$

(and coincides with structure in the big cartesian closed category)

GAMES FOR LOGIC

(Dialogue games: Lorenzen)

IDEA (in the $\wedge$ $\rightarrow$ fragment)

- Faced with a $\wedge$

  O challenges us to prove one of the conjuncts

- Faced with a $\rightarrow$

  P either disputes the hypothesis or accepts the conclusion

And P can backtrack

COLOURED GAMES

The moves in A are coloured
p, q, r ... by propositional letters.

The game $\bullet p$ with single move
coloured p represents p.

Direct innocent strategies are
innocent strategies in which we
always respond to a move with
a move of same colour.

FULL ABSTRACTION

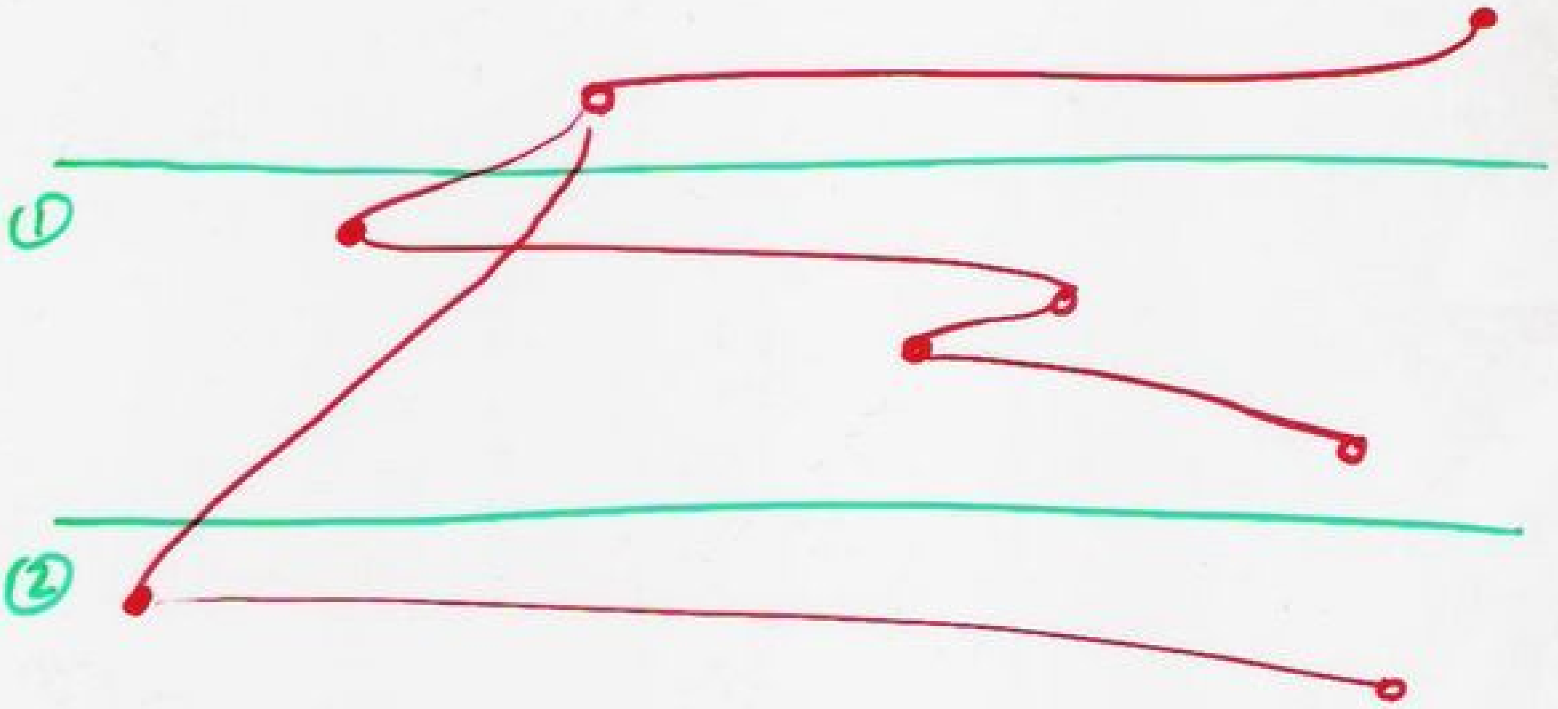
Finite total direct innocent strategies

~ Typed λ-calculus

~ Closed cartesian (multi)category
freely generated

EXAMPLES

$$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$$



$$\lambda f g x .\ f x (g x)$$

$$((p \to p) \to p) \to p$$

$$\lambda F .\ F(\lambda x .\ F(\lambda y .\ x))$$

GAMES WITH AXIOM LINKS

Coloured games as before but now have strategies with explicit information as to which previous O-occurrence of p P relies on when playing p.

EXAMPLE

$$((p \longrightarrow q) \longrightarrow p) \longrightarrow p$$

FULL ABSTRACTION (Olivier Laurent)

Innocent strategies with links

$\sim$ Typed $\lambda\mu$ calculus

$\sim$ Control (multi)category

OBSERVATIONS

- Composition of innocent
  - huge generalization of λ-calculus
  - different combinatorics
- Abstract machines
- Applications to syntax
  - properties of PCF
  - type isomorphism

One of many subtle compositions?