CONSTRUCTING HYPERDOCTRINES

Towards a Model Theory of Type Theory

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GÖDEL'S DIALECTICA INTERPRETATION

Gödel 1942: Princeton Lectures

Proof theory:

functional interpretation

Consistency of number theory

analysis

Proof mining (Kohlenbach)

SIGMA - PI

Focus: quantifier combination

$$\exists u \in U.\ \forall x \in X.\ \square$$

so for type theory

$$\Sigma u \in U\ \Pi x \in X\ \square$$

i.e. SUM - PRODUCT

DIALECTICA CATEGORIES

1980s (de Paiva's thesis: Robinson
Rosolini)

Abstract view of $\longrightarrow$

$$\begin{tikzcd} U \arrow[d] & X \arrow[l, "A"', "|" marking] \\ V & Y \arrow[l, "|" marking] \end{tikzcd}$$

$f : U \Longrightarrow V$

$F : U \times Y \Longrightarrow X$

$$\phi : \Pi_{uy}\, A(u, F(u,y)) \Longrightarrow B(f(u), y)$$

VARIATIONS

1980s Diller-Nahm (1974) and Linear Logic

2000s Copenhagen (Biering, Birkedal, H. Rossini, Shreider, van Oosten)

2010s Explosion of interest

In CS

Plotkin et al The Computer Fried

Fiore, Olivia, Powell, Robinson

(And now French students)

LOGIC TO DEPENDENT TYPES

Is there a Dialectica-style interpretation of Dependent Type Theory?

YES!

Moreover Category Theory is key to the understanding.

AIM OF TALK

Explain

Given a categorical model $\mathbb{E}$ of type theory there is a model $\Sigma\Pi(\mathbb{E})$, a variant of Dialectica.

Method

From $\mathbb{E}$ construct a new model $\mathrm{Poly}(\mathbb{E})$ and $\Sigma\Pi(\mathbb{E}) \doteq \mathrm{Poly}^2(\mathbb{E})$.

PLAN OF TALK

- Overview of categorical models
- Sketch of the polynomial model (von Glehn)
- Concrete details on the baby example.
- Remarks on Dialectica and beyond.

CATEGORICAL SETTING

(but see Awodey's talk)

Category $\mathbb{C}$ with collection of exceptional maps $\mathbb{E}$ (display maps, fibrations ...) with

$\mathbb{E}$ cartable

(and with all maps $C \to 1$ in $\mathbb{E}$)

So a fibration $\mathbb{E} \to \mathbb{C}$ of restricted kind.

(Coherence suppressed.)

SUMS

$\mathbb{E} \to \mathbb{C}$ has strong sums in the sense that $\mathbb{E}$ is closed under composition

So left adjoints to pb along maps in $\mathbb{E}$. (Beck-Chevalley automatic.)

Joyal's notion of tribe

(Already theory at this level.)

PRODUCTS

$\mathbb{E} \to \mathbb{C}$ has products in the sense that we have right adjoints to the pb of $\mathbb{E}$ maps along $\mathbb{E}$ maps.

(Beck - Chevalley inherited from sums.)

Joyal's notion of π-tribe

FACTORIZATION

(for Identity Types & whatever)

We have a functorial factorization of maps $\xrightarrow{f}$ in $\mathbb{C}$ as

$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "a"'] & & \cdot \\ & \cdot \arrow[ur, "e"'] & \end{tikzcd}$$

with a anodyne i.e. has llp with respect to $\mathbb{E}$ maps; and moreover $\mathbb{E}$ pbs of anodyne are anodyne

$\sim$ Joyal's notion of h-trivial.

COPRODUCTS

Special property amounting to strong rules for (some) finite inductive types.

$\mathbb{E} \to \mathbb{C}$ has coproducts in each fibre, stable under pb.

(Possible weakenings of this but not for now.)

MAXIMAL CASE

$\mathbb{E} = \mathbb{C}$

Factorization is trivial

Result is a locally cartesian closed category with coproducts

[This goes back to Seely.]

MINIMAL CASE

$\mathbb{E}$ is the collection of product projections.

The factorization is

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "{(id_A, f)}"'] & & B \\ & A \times B \arrow[ur, "snd"'] & \end{tikzcd}$$

The result is a cartesian closed category with coproducts. (Folklore.)

MODEL THEORY OF TYPE THEORY

~ Relative Interpretation

Cf 40+ years experience with Topos Theory

- Construction of new models from old
- Study of the properties of models

POLYNOMIALS / CONTAINERS

(In the fibre over I)

objects $\quad U \leftarrow X$

maps

$$\begin{tikzcd}
V \arrow[d, "f"'] & Y \arrow[l] & f^*X \arrow[l, "F"'] \arrow[d] \\
U & & X \arrow[ll]
\end{tikzcd}$$

Huge literature: Gambino – Kock

Kock^2

Abbott, Altenkirch, Ghani

FUNDAMENTAL OBSERVATION

Altenkirch, Levy, Staton: Over <u>Sets</u>

the category of polynomials is cartesian closed.

Hyvernat The proof can be written in type theory so as to apply to any locally cartesian closed category with coproducts.

MAIN THEOREM

(von Glehn) For any model $\mathbb{E} \longrightarrow \mathbb{C}$ of type theory, there is a model

$$\mathrm{Poly}(\mathbb{E}) = \mathrm{Poly}(\mathbb{E} \longrightarrow \mathbb{C})$$

with underlying category polynomials $U \longleftarrow X$ from $\mathbb{E}$ and with displays

$$\begin{tikzcd}
V \arrow[d, "\in \mathbb{E}"'] & Y \arrow[l] & f^*X \arrow[l, "\text{coproduct injection}"'] \arrow[d] \\
U & & X \arrow[ll]
\end{tikzcd}$$

OUTLINE

By A-L-S

$$
\begin{tikzcd}
\mathbb{E} \arrow[rr, shift left] \arrow[rr, phantom, "\top"] \arrow[dr] & & \mathrm{Poly}(\mathbb{E}) \arrow[ll, shift left] \arrow[dl] \\
& \mathbb{C} &
\end{tikzcd}
$$

Day situation for the cccs, and with Σs Πs

Extend
$$
\begin{tikzcd}
\mathrm{Poly}(\mathbb{E}) \arrow[d] \\
\mathbb{C}
\end{tikzcd}
\quad \text{to} \quad
\begin{tikzcd}
\mathrm{Poly}(\mathbb{E}) \arrow[d] \\
\mathrm{Poly}(\mathbb{C})
\end{tikzcd}
$$

defining factorization and then extend Σ and Π

FUNCTIONAL EXTENSIONALITY

Characterizes the identity on function spaces

$$\prod_{a \in A} \mathrm{Id}_{B(a)}(f(a), g(a)) \longrightarrow \mathrm{Id}_{\prod_{a \in A} B(a)}(f, g)$$

Joyal: $\prod$ preserves the homotopy relation

Univalence $\Longrightarrow$ Functional extensionality

OBSERVATION

(von Glehn)

Functional extensionality fails in non-trivial polynomial models.

Conclusion

A model theory of type theory should NOT focus on univalent models.

UNIVERSAL PROPERTY

$$\text{"}\mathrm{Poly}(\mathbb{E}) = \Sigma(\mathbb{E}^{op})\text{"}$$

sums along $\mathbb{E}$ maps

(Folklore but see recent papers

Hofstra

Gambino - Kock

Kock - Kock

and more)

EXAMPLE : FINITE SETS

What is

$$\mathrm{Poly}(\mathbb{F}) = \Sigma(\mathbb{F}^{op})$$

finite sets

As a category it is finite sums of representables in the object classifier

$$[\mathbb{F}_1\ \mathrm{Set}].$$

THE GENERIC OBJECT

$$U = \mathbb{F}(1,-) \quad : k \longmapsto k$$

$$U^m = \mathbb{F}(m,-) : k \longmapsto k^m$$

So $Poly(\mathbb{F})$ is the full subcategory on objects of the form

$$\sum_i \mathbb{F}(n_i,-) = \sum_i U^{n_i}$$

ATOMS (or TINY OBJECTS)

Objects A such that the right adjoint $A \Rightarrow -$ to $A \times -$ has itself a right adjoint.

In $[\mathbb{C}^{op}, \mathrm{Set}]$ a representable $\mathbb{C}(-, d)$ is an atom iff

$\mathbb{C}(-, d) \Rightarrow -$ preserves colimits

iff

for all $c \in \mathbb{C}$ the product $c \times d$ is in the idempotent completion.

CARTESIAN CLOSURE

Closure under finite products is evident.

Function spaces follows from

BASIC OBSERVATION (Lawvere)

$$U \Rightarrow U \cong 1 + U$$

plus atomicity.

DISPLAYS / FIBRATIONS

Maps $p : J \to I$

$$\begin{array}{c} \sum_{j \in J} \mathbb{F}(m_j, -) \\ \downarrow \\ \sum_{i \in I} \mathbb{F}(n_i, -) \end{array}$$

and for each j

$$n_{p(j)} \xrightarrow[\phi_j]{} m_j$$

where each ϕ_j is a coproduct injection.

i.e.

$$\begin{array}{c} \sum_{j \in J} U^{m_j} \\ \downarrow \\ \sum_{i \in I} U^{n_i} \end{array}$$

$p : J \to I$

$\pi_j : U^{m_j} \to U^{n_{p(j)}}$ a projection.

FACTORIZATION

$$\begin{tikzcd}
\sum_{j\in J} U^j \arrow[rr, "{(p,\phi)}"'] \arrow[dr] & & \sum_{i\in I} U^i \\
& \sum_{j\in J} U^j \times U^{p(j)} \arrow[ur] &
\end{tikzcd}$$

is given by summing the minimal factorization.

IDENTITY

For the atom U we have

$$\prod_{u,v \in U} Id_U(u,v)$$

is true.

So for general $\Sigma_i U^i$ we have

$$\Sigma_i U^i \times U^i \longrightarrow \Sigma_{i,i'} U^i \times U^{i'}$$

EXTENSIONALITY

For any $A \in Poly(\mathbb{F})$ we must have

$$\prod f, g : A \Rightarrow U . \prod a \in A . \, Id_U(f(a), g(a))$$

is true.

However

$$Id_{U \Rightarrow U} = Id_{1+U}$$

is not universally true.

Failure of functional extensionality

A DIALECTICA INTERPRETATION

Given $\mathbb{E} = (\mathbb{E} \longrightarrow \mathbb{C})$ a model for type theory we have $\mathrm{Poly}(\mathbb{E})$ a model for type theory: so iterate $\mathrm{Poly}^2(\mathbb{E})$?

Almost

$$\mathrm{Poly}^2(\mathbb{E}) = \sum \left(\sum \mathbb{E}^{op} \right)^{op}$$

$$= \sum \prod \mathbb{E} \quad \text{a Dialectica Interpretation}$$

(Correct if we stay with fibrations over $\mathbb{C}$ until the last step.)

FUNCTORS

(Polynomial case)

① $\mathbb{E} \longrightarrow \mathrm{Poly}(\mathbb{E})\ ;\ U \longmapsto (U \leftarrow 0)$

analogue of the constant functor

$$\mathrm{Set} \longrightarrow [\mathbb{C}^{op}, \mathrm{Set}]$$

② $\mathbb{E}^{op} \longrightarrow \mathrm{Poly}(\mathbb{E})\ ;\ X \longmapsto (1 \leftarrow X)$

analogue of the Yoneda

$$\mathbb{C} \longrightarrow [\mathbb{C}^{op}, \mathrm{Set}]$$

FUNCTORS

(Dialectica case)

We get two functors

$$\mathbb{E} \longrightarrow \Sigma\Pi\,\mathbb{E}$$

①

$$\mathbb{E} \longrightarrow \mathrm{Poly}(\mathbb{E}) \longrightarrow \mathrm{Poly}(\mathrm{Poly}\,\mathbb{E})$$

②

$$\mathbb{E} \longrightarrow (\mathrm{Poly}(\mathbb{E}))^{op} \longrightarrow \mathrm{Poly}(\mathrm{Poly}\,\mathbb{E})$$

AXIOM OF CHOICE

In models for type theory we have

$$\Pi a \in A. \Sigma b \in B(a).\ C(a,b)$$

$$\longrightarrow$$

$$\Sigma f \in \Pi a \in A. B(a).\ \Pi a \in A.\ C(a, f(a)).$$

an isomorphism (inverse to the obvious map)

So hereditarily everything in type theory has $\Sigma\Pi$ form.

INTERPRETATION FUNCTOR

AC provides a distributive law

$$\Pi\Sigma \longrightarrow \Sigma\Pi$$

and so $\Sigma\Pi$ is a pseudo monad.

Each model $\mathbb{E} \to \mathbb{C}$ is an algebra for $\Sigma\Pi$ i.e. we get an interpretation

$$\Sigma\Pi\mathbb{E} \longrightarrow \mathbb{E}$$

Both functors $\mathbb{E} \longrightarrow \Sigma\Pi\mathbb{E}$ are sections.

(Glimpse of a theory.)

INDEPENDANCE OF PREMIS

Gödel's Interpretation validates

$$\left(\forall x\, \phi(x) \longrightarrow \exists y\, \psi(y)\right) \to \exists y\left(\forall x.\, \phi(x) \longrightarrow \psi(y)\right)$$

where ϕ is QF.

In $\Sigma\Pi E$

$$\Pi x \in X.\, A(x) \Longrightarrow \Sigma v \in V.\, \Pi y \in Y(v).\, B(y)$$

$$\cong \Sigma v \in V.\left(\Pi x \in X.\, A(x) \Rightarrow \Pi y \in Y(v).\, B(y)\right).$$

What is the connection?

CAUTION

Much to understand:

- model theory of type theory is in its infancy
- functorial interpretation gives many more models
- there are parallels (extensionality, IP) between $\Sigma\Pi\mathrm{E}$ and Gödel's interpretation but hard to make precise.