LINEAR

LOGIC

CLASSICAL LINEAR LOGIC

- Separates out the complexities arising from the rules of weakening, contraction in classical sequent calculus

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, B}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, B, B}{\Gamma \vdash \Delta, B}$$

- Classical in the sense that it maintains the symmetry of classical sequent calculus.

RULES FOR CONJUNCTION (and DISJUNCTION)

Conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

× ve form

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

+ ve form

Disjunction

dual to the above

SYNTAX of CLL

(Propositional fragment only)

Multiplicatives

Conjunctions I , $\otimes$

Disjunctions $\perp$, $⅋$

Exponentials ('Modalities')

"Of course" $!$ "Potentially infinitary tensor"

"Why not" $?$ "Potentially infinitary par"

Additives

Conjunctions $\top$, $\&$

Disjunctions 0 , $\oplus$

Involution

Negation $(\ )^{\perp}$

BASIC RULES

AXIOM

$$\frac{}{A \vdash A}$$

CUT

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

EXCHANGE

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, C, D, \Delta'}{\Gamma \vdash \Delta, D, C, \Delta'}$$

NEGATION

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta} \qquad \frac{\Gamma \vdash C, \Delta}{\Gamma, C^\perp \vdash \Delta}$$

MULTIPLICATIVE RULES

Conjunctions

$$\frac{}{\vdash I} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, I \vdash \Delta}$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}$$

Disjunctions

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta} \qquad \frac{}{\bot \vdash}$$

$$\frac{\Gamma \vdash C, D, \Delta}{\Gamma \vdash C \parr D, \Delta} \qquad \frac{\Gamma, C \vdash \Delta \qquad \Gamma', D \vdash \Delta'}{\Gamma, \Gamma', C \parr D \vdash \Delta, \Delta'}$$

EXPONENTIAL RULES

'Of course'

(Comonad rules)

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad \frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}$$

(Comonoid rules)

$$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}$$

'Why not'

(Par monoid rules)

$$\frac{\Gamma \vdash C, \Delta}{\Gamma \vdash ?C, \Delta} \qquad \frac{!\Gamma, C \vdash ?\Delta}{!\Gamma, ?C \vdash ?\Delta}$$

(Par comonoid rules)

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash ?C, \Delta} \qquad \frac{\Gamma \vdash ?C, ?C, \Delta}{\Gamma \vdash ?C, \Delta}$$

ADDITIVES

Skip the rules!!! It should be enough to know that eventually they give proof theoretic equivalences

$$!\top \cong I$$

$$!(A \times B) \cong\ !A \otimes !B$$

$$?0 \cong \bot$$

$$?(A \oplus B) \cong\ ?A \parr ?B$$

EXAMPLE

(Weak distributivity)

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \otimes B} \qquad C \vdash C}{A,\ B \parr C \vdash A \otimes B,\ C}}{A,\ B \parr C \vdash (A \otimes B) \parr C}}{A \otimes (B \parr C) \vdash (A \otimes B) \parr C}
$$

NOTE Two 'irrelevent' choices

$\Downarrow$

Proof Nets

NOTE While $A \otimes (B \parr C) \vdash (A \otimes B) \parr C$ is canonically provable, there can be no cut-free (& hence no) proof of $(A \otimes B) \parr C \vdash A \otimes (B \parr C)$

A MANY-FEATURED PROOF

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{!A \vdash A}}{!A, A^\perp \vdash}}{!A, ?A^\perp \vdash} \qquad B \vdash B}{!A \quad (?A^\perp \parr B) \vdash B}}{!A \quad !(?A^\perp \parr B) \vdash B}}{!A, !(?A^\perp \parr B) \vdash !B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{!A \vdash A}}{!A, A^\perp \vdash}}{!A, ?A^\perp \vdash} \qquad ?B^\perp \parr C \vdash ?B^\perp \parr C}{!A, (?A^\perp \parr (?B^\perp \parr C)) \vdash ?B^\perp \parr C}}{!A, !(?A^\perp \parr (?B^\perp \parr C)) \vdash ?B^\perp \parr C}}{!A, !(?A^\perp \parr (?B^\perp \parr C)) \vdash !(?B^\perp \parr C)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \vdash B}{!B \vdash B}}{!B, B^\perp \vdash}}{!B, ?B^\perp \vdash} \qquad C \vdash C}{!B, (?B^\perp \parr C) \vdash C}}{!B, !(?B^\perp \parr C) \vdash C}$$

$$\dfrac{!A, !(?A^\perp \parr (?B^\perp \parr C)) \vdash !(?B^\perp \parr C) \qquad !B, !(?B^\perp \parr C) \vdash C}{!A, !(?A^\perp \parr (?B^\perp \parr C)), !B \vdash C}$$

$$\dfrac{\dfrac{!A, !(?A^\perp \parr B) \vdash !B \qquad !A, !(?A^\perp \parr (?B^\perp \parr C)), !B \vdash C}{!A, !A, !(?A^\perp \parr B) \quad !(?A^\perp \parr (?B^\perp \parr C)) \vdash C}}{!A, !(?A^\perp \parr B), !(?A^\perp \parr (?B^\perp \parr C)) \vdash C}$$

$$\dfrac{A, (A \to B) \vdash B \qquad \dfrac{A, A \to (B \to C) \vdash B \to C \qquad B, B \to C \vdash C}{A, A \to (B \to C), B \vdash C}}{A, (A \to B) \quad A \to (B \to C) \vdash C}$$

SOME DUALITIES

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A^\perp, A} \qquad \dfrac{B \vdash B}{\vdash B^\perp, B}}{\vdash A^\perp, B^\perp, (A \otimes B)}}{(A \otimes B)^\perp \vdash A^\perp, B^\perp}}{(A \otimes B)^\perp \vdash A^\perp \parr B^\perp}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, A^\perp \vdash} \qquad \dfrac{B \vdash B}{B, B^\perp \vdash}}{A, B, A^\perp \parr B^\perp \vdash}}{A \otimes B, \; A^\perp \parr B^\perp \vdash}}{A^\perp \parr B^\perp \vdash (A \otimes B)^\perp}$$

"SO" $(A \otimes B)^\perp \cong A^\perp \parr B^\perp$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A, A^\perp}}{\vdash A, ?A^\perp}}{\vdash !A, ?A^\perp}}{(!A)^\perp \vdash ?A^\perp}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, A^\perp \vdash}}{!A, A^\perp \vdash}}{!A, ?A^\perp \vdash}}{?A^\perp \vdash (!A)^\perp}$$

"SO" $(!A)^\perp \cong ?A^\perp$

ONE-SIDED CALCULUS

(Modulo dualities)

Axiom $$\frac{}{\vdash A, A^{\perp}}$$

Cut $$\frac{\vdash A, \Gamma \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}$$

Exchange $$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}$$

Multiplicatives

$$\frac{}{\vdash I} \qquad\qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \qquad\qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}$$

Exponentials

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma} \qquad\qquad \frac{\vdash A, ?\Delta}{\vdash !A, ?\Delta}$$

$$\frac{\vdash \Gamma}{\vdash ?A\ \Gamma} \qquad\qquad \frac{\vdash ?A, ?A, \Delta}{\vdash ?A, \Delta}$$

Additives – no room!

INTUITIONISTIC LINEAR LOGIC

- Asymmetric system
- Good for many purposes
  (eg. foundations of functional programming)
- Lots of naturally occurring 'models'.

Syntax of ILL

Multiplicatives $\quad I \,,\ \otimes \,,\ \multimap$

Exponential $\quad !$

Additives $\quad \top \,,\ \&$

USES: Single conclusion sequents

SEQUENT RULES for ILL

Axiom $$\frac{}{A \vdash A}$$

Cut $$\frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}$$

Exchange $$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

Multiplicatives

$$\frac{}{\vdash I} \qquad \frac{\Gamma \vdash A}{\Gamma, I \vdash A}$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}$$

Exponential

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$$

Additives again no room

GIRARD TRANSLATION OF INTUITIONISTIC LOGIC (very good for minimal logic)

Intuitionistic logic $\qquad\qquad$ ILL

$$A \longmapsto t(A)$$

ATOMS $\qquad A \longmapsto A$

RECURSIVE CLAUSES $\qquad 1 \longmapsto \top$

$$A \wedge B \longmapsto t(A) \,\&\, t(B)$$

$$A \to B \longmapsto\ !t(A) \multimap t(B)$$

TRANSLATION OF DEDUCTIONS

$$\Gamma \vdash A \longmapsto\ !t(\Gamma) \vdash t(A)$$

CORRECTNESS OF THE TRANSLATION

Essentially require to show

$$\frac{A \wedge B \vdash C}{\overline{A \vdash B \to C}}$$

is reflected by the translation.

But

$$\begin{array}{ll}
!\,(A \wedge B)^t \vdash C^t & \\
\hline\hline
!\,(A^t \,\&\, B^t) \vdash C^t & \text{equality} \\
\hline\hline
!A^t \otimes !B^t \vdash C^t & \text{by an isomorphism} \\
\hline\hline
!A^t \vdash\, !B^t \multimap C^t & \text{by ILL} \\
\hline\hline
!A^t \vdash (B \to C)^t & \text{equality}
\end{array}$$

BIG DISCOVERY

(especially in view of LC, LU, ...)

More usual logic can be decomposed into logic-like elements which correspond to the 'propositions' of linear logic.

- How to think of these 'micro-propositions'?

- In what way to think of the world of propositions as standing within this wider world?

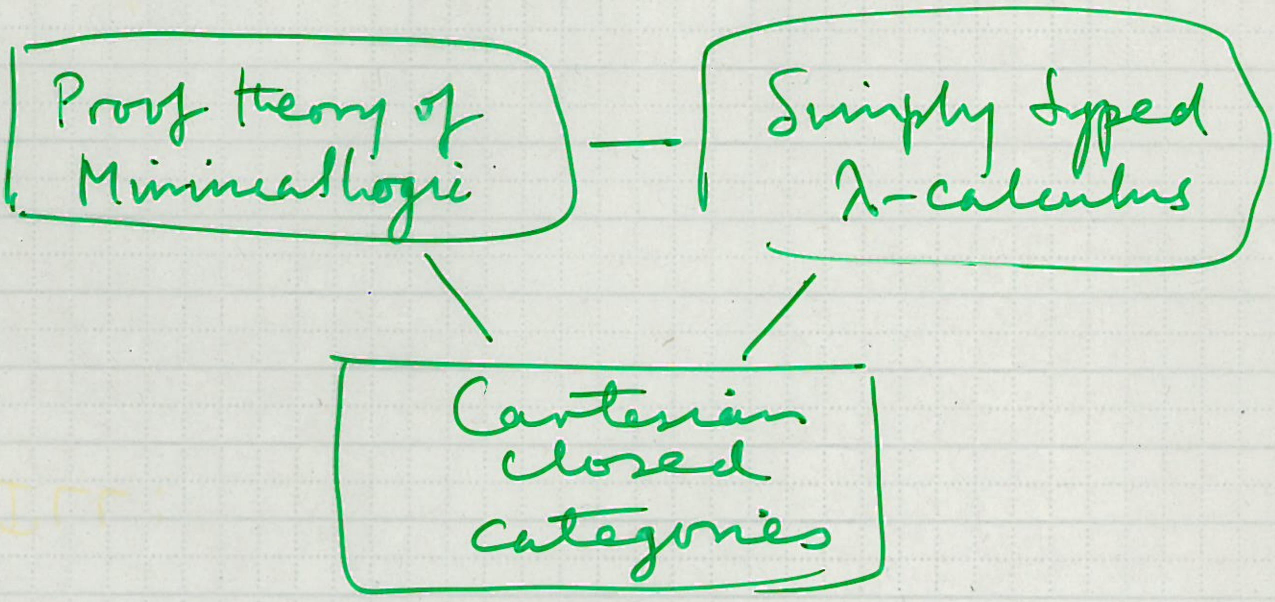
The idea does (computer science) work.

Does it make philosophical sense?

MODELS
via
CURRY – HOWARD

(i.e. after identification of cut-free proofs)

COMPARE



CATEGORICAL MODELS OF CLL/ILL

For CLL:

Have a $*$-autonomous category plus a comonad s.t. (plus products).

For ILL:

Have a symmetric monoidal closed category plus a comonad s.t. (plus products).

Examples

CLL V-lattices, coherence spaces

ILL Vector spaces, domains & linear maps

(Also assorted categories of games.)

DUALITY CONSTRUCTIONS

(An example)

Suppose $\mathbb{C}$ is smcc with products;
then $\mathbb{C} \times \mathbb{C}^{op}$ is $*$-autonomous.

Enough to define I, $\otimes$, $(\ )^{\perp}$ on $\mathbb{C} \times \mathbb{C}^{op}$ & check it works:-

I is $(I, \top)$

$$(U, X) \otimes (V, Y) = \big(U \otimes V, (V \multimap X) \times (U \multimap Y)\big)$$

$$(U, X)^{\perp} = (X, U)$$

REPRESENTATION OF PROOFS

SEEK Canonical representation of (equivalence classes of) normal proofs

$\Downarrow$

CONCRETE presentation of the free s&s category

(*Maybe* good for applications.)

Suitable such for linear Logic are

'Proof Nets'

PROOF NETS for CLL
(Multiplicative fragment)

Graph with links as follows :-

Axiom $\quad \overline{A \qquad A^{\perp}}$

Cut $\quad \underline{A \qquad A^{\perp}}$

Tensor $\quad \dfrac{A \qquad B}{A \otimes B}$

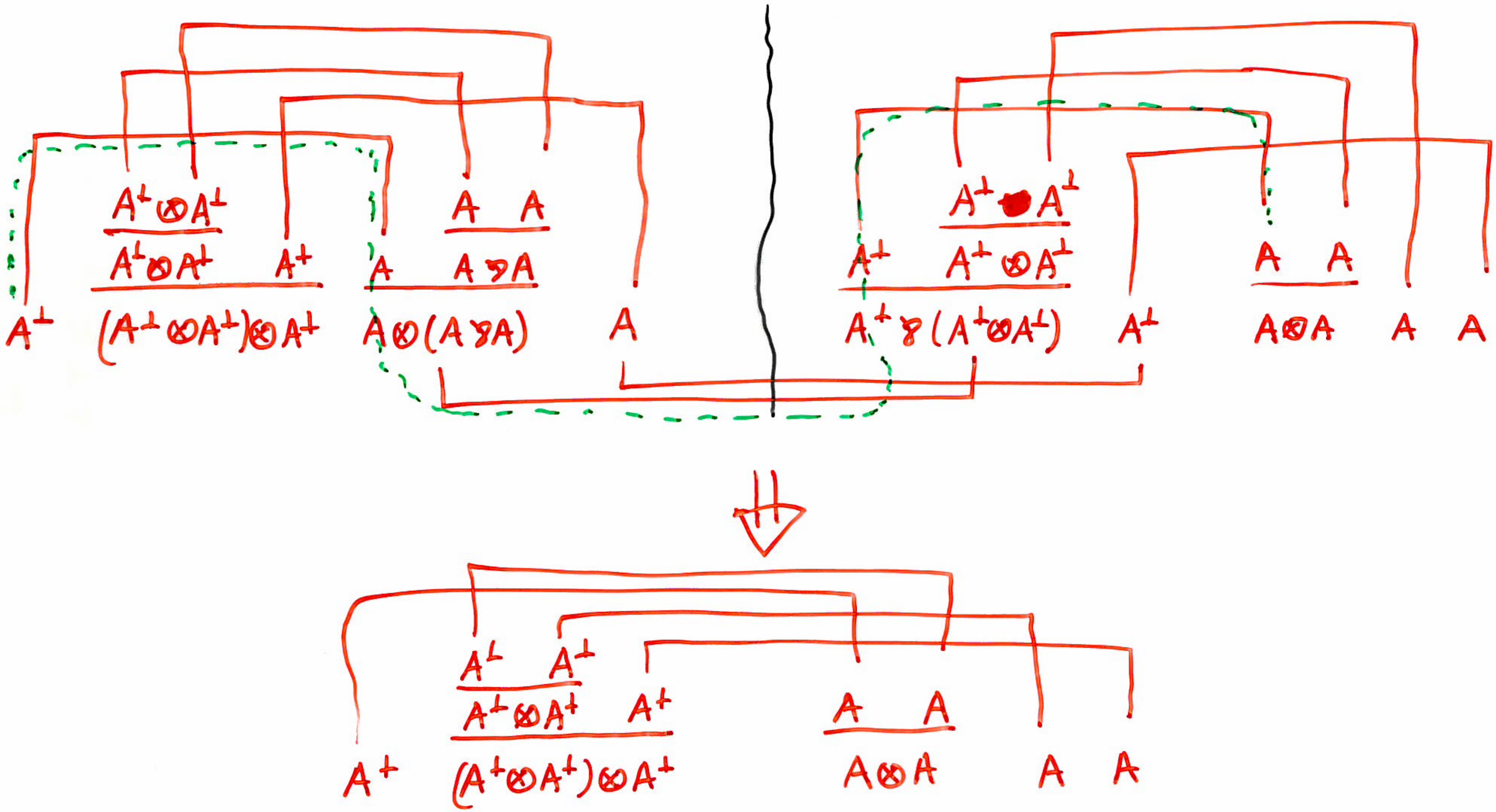
Par $\quad \underbrace{\dfrac{A \qquad B}{A \parr B}}$

A switching chooses one or other of these

CORRECTNESS CRITERION A graph (without cut) represents (an equivalence of)/a proof iff for every switching (Danos – Regnier) it is a 'tree' i.e. it is connected acyclic.

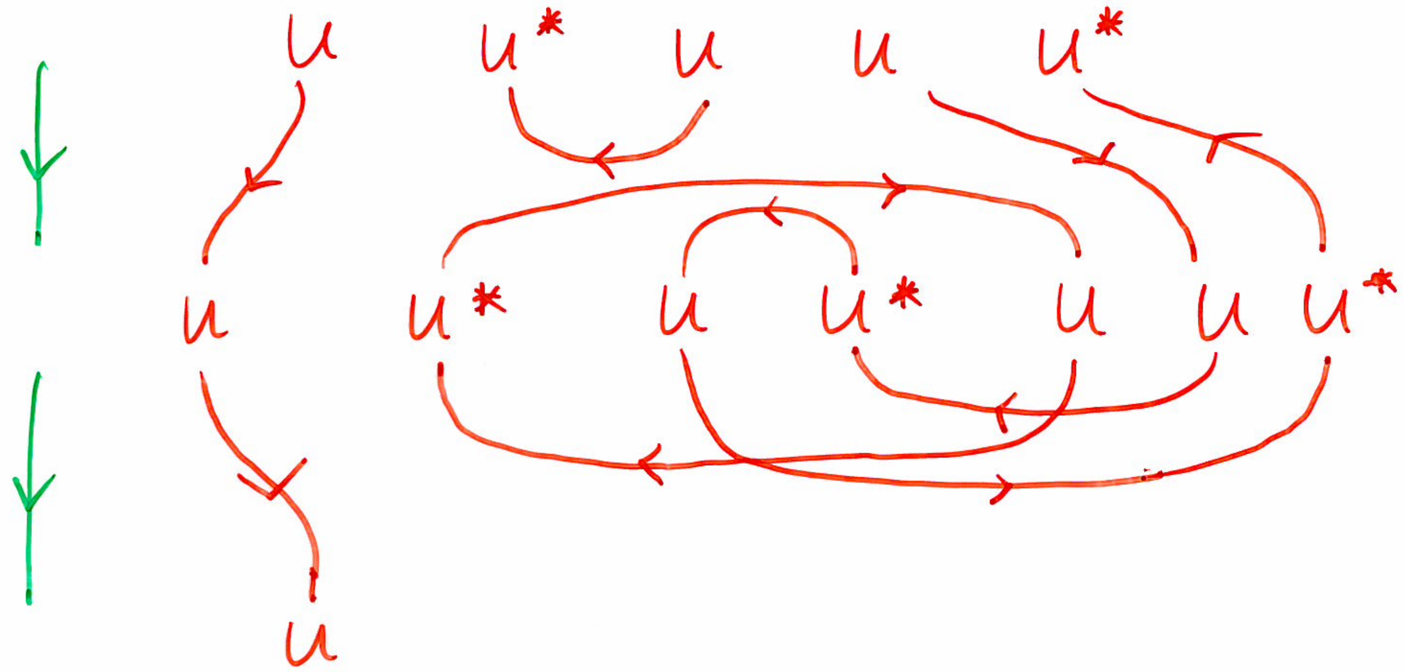
'TURBO' CUT ELIMINATION

(Connect the wires!)



THE FREE 'COMPACT CLOSED' CATEGORY

(Degenerate case when $\otimes = \wp$)



I.e. Linear logic with no correctness criterion

Full Completeness Thms

Assume a $*$-autonomous $\mathbb{C}$.

If $\Gamma(A_1, \ldots A_n)$ in the $\times$ ve fragment, then for each interpretation $[\![A_i]\!] \in \mathbb{C}$, we have $[\![\Gamma(A_1 \ldots A_n)]\!] \in \mathbb{C}$.

A proof $\vdash \Gamma(A_1, \ldots, A_n)$ in CLL gives a $\boxed{\text{uniform}}$ map

$$p_{[\![\,]\!]} : I \longrightarrow [\![\Gamma(A_1, \ldots A_n)]\!] \in \mathbb{C}$$

$\boxed{\text{QUESTION Do such (non-trivial) uniform maps correspond exactly to proofs?}}$

"Schur-Weyl Duality"

(Generalizes: Which matrices commute with all matrices?)

Let V be a finite dimensional vector space of dimension ≥ 2, and $\mathbb{C}$ the compact closed subcategory of finite dimensional vector spaces generated by V.

Now restrict to $\overline{\mathbb{C}}$ whose maps are uniform w.r.t. the action of $GL(V)$.

Then $\overline{\mathbb{C}}$ is the free k-linear compact closed category on a single object.

ABSTRACT GEOMETRY OF INTERACTION

Based on (certain) symmetric monoidal categories with trace

$$\boxed{\text{Trace} \quad = \quad \text{Feedback loop}}$$

$$\Downarrow$$

(Special) compact closed categories.

Interpret CLL in degenerate models so

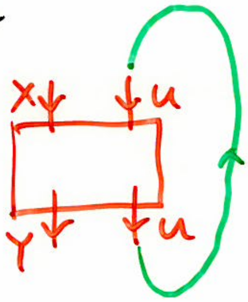
Cut $\longleftrightarrow$ Contraction of indices

Extract computational information!

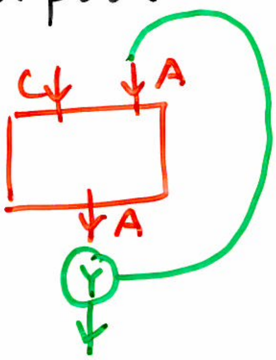
CARTESIAN CATEGORIES WITH TRACE

For a category with $\times$s, a trace is equivalent to a (goodish) fixed point operator.

Trace



Fixed points



TOY EXAMPLE

<u>Objects</u> Pairs of (finite) sets

$$(U, X)$$

<u>Maps</u> $(U, X) \longrightarrow (V, Y)$

Partial maps

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \\ V & Y \arrow[l] \arrow[u] \end{tikzcd}$$

$U \longrightarrow X + V$

$Y \longrightarrow X + V$

<u>Composition</u>