ALGEBRA & DIAGRAMS in the LAMBDA CALCULUS

for
Samson Abramsky
against
philistine prejudice

"Category Theory cannot really have anything to do with Computer Science"!

ALGEBRAIC THEORIES

$T : \mathbb{F} \longrightarrow Set$ with

$pr_i \in T(n)$

$T(n) \times T(m)^n \longrightarrow T(m)$

natural and with identity & associativity rules

N.B. If $U \in \mathbb{C}$ with products then $\mathbb{C}(U^n, U)$ is the endomorphism theory of U

T - ALGEBRAS

$\mathrm{Alg}(T)$ $\quad T(n) \times A^n \longrightarrow A$

- $T(m)$ is the free T-algebra on m generators.

- For any A there is a theory T_A of T-algebra extensions of A

PRESHEAVES ON T

PT $\qquad X(n) \times (T(m))^n \longrightarrow X(m)$

- $U = (T(m))_m$ is the universal object
- (Yoneda)

$$PT(U^n, U) \cong T(n)$$

as algebraic theories

- (Lawvere)

$$U \Rightarrow U(n) \cong T(n+1)$$

LAMBDA THEORIES

A λ-theory $\mathcal{L}$ is a semi-closed algebraic theory i.e.

$\mathcal{L}(n+1) \lhd \mathcal{L}(n)$ natural and respecting composition

$$\mathcal{L}(n) \longrightarrow \mathcal{L}(n+1)\, ; \quad s \longmapsto s.\underset{\substack{\uparrow \\ \text{new variable}}}{z}$$

$$\mathcal{L}(n+1) \longrightarrow \mathcal{L}(n)\, ; \quad t \longmapsto \lambda z.t$$

- Clear how to interpret the λ-calculus (almost no other way).

- $\Lambda(n) = (\lambda\text{-terms in } n)/\beta$ is the initial λ-theory.

CATEGORICAL MODELS

Suppose $U^U \lhd U$ in $\mathbb{C}$ with products. Then

$$\mathbb{C}(U^n, U) \text{ is a } \lambda\text{-theory}$$

Representation Theorem

For any $\mathcal{L}$, $U^U \lhd U$ in $P\mathcal{L}$

the Yoneda isomorphism

$$P\mathcal{L}(U^n, U) \cong \mathcal{L}(n)$$

is an isomorphism of λ-theories

SCOTT – TAYLOR THEOREM

In PC the indexed category of retracts of U has Πs, Σs along its fibred maps

Corol (Paul Taylor) The category of retracts is relatively cartesian closed

Corol (Dana Scott) The category of retracts is cartesian closed

Λ - ALGEBRAS

Λ the initial λ-theory has algebras

$$\Lambda(n) \times A^n \longrightarrow A$$
$$t(\underline{x}), \underline{a} \longmapsto [\![t(\underline{a})]\!]_A$$

- Λ-algebras $\simeq$ λ-algebras
- No direct proofs of semantic examples
- But $\Lambda \longrightarrow \mathcal{L}$ gives $\mathcal{L}(0)$ (the initial $\mathcal{L}$-algebra) the structure of a Λ-algebra.

Λ-ALGEBRAS to λ-THEORIES

OUTLINE

Λ-algebra $A \longmapsto$ "λ-monoid" $A(1)$

$\longmapsto U^U \lhd U$ in $PA = PA(1)$

$\longmapsto \mathcal{U}_A(n) = PA(U^n, U)$ corresponding λ-theory.

(Functoriality is not trivial.)

PRESHEAVES on the MONOID

$$\mathbb{1} = \lambda xy.xy$$

$$A(\mathbb{1}) = \{a \in A \mid \mathbb{1}a = a\}$$
$$= \{a \in A \mid \lambda x.ax = a\}$$

Unit $\quad I = \lambda x.x \quad \in A(\mathbb{1})$

Composition $\quad a, b \longmapsto \lambda x.a(bx);$

$$A(\mathbb{1}) \times A(\mathbb{1}) \longrightarrow A(\mathbb{1})$$

The universal $\mathcal{U} \in PA$ is $A(\mathbb{1})$ with right action by composition

Take

$$A(2) = \{d \in A(\mathbb{1}) \mid \mathbb{1} \circ d = d\} \triangleleft A(\mathbb{1})$$
$$= \{d \in A \mid \lambda xy.dxy = d\}$$

with right action by composition

FACT $\quad A(2) \cong \mathcal{U}^{\mathcal{U}}$

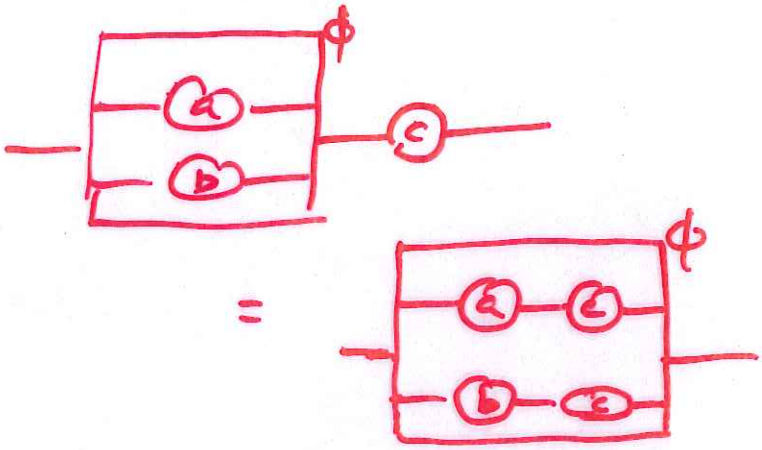
A KEY POINT

U^U has underlying set

$$PA(U \times U, U)$$

the equivariant maps

Diagram:



FACT Any $\phi \in PA(U \times U, U)$ is uniquely of form

$$(a, b) \longmapsto \lambda x.\, d(ax)(bx)$$

$$d \in A(2)$$

ALGEBRAIC STRUCTURE

$A(1) \times A(1) \longrightarrow A(1)$

$a, b \longmapsto \lambda x.\, a(bx)$

$A(2) \times A(1)^2 \longrightarrow A(1)$

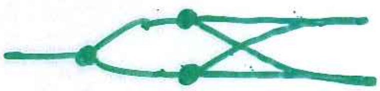
$d, a, b \longmapsto \lambda x.\, d(ax)(bx)$

$A(1) \times A(2) \longrightarrow A(2)$

$a, d \longmapsto \lambda xy.\, a(dxy)$

$A(2) \times A(2)^2 \longrightarrow A(2)$

$d, e, f \longmapsto \lambda xy.\, d(exy)(fxy)$



$I \in A(1) \quad T = \lambda xy.x \; , \; F = \lambda xy.y \in A(2)$

unit and associative laws

PLUS $\quad p = \lambda x.xT \in A(1)$

$q = \lambda x.xF \in A(1)$

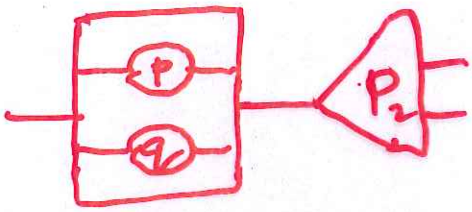
$P_2 = \lambda yzx.\; xyz \in A(2)$

with $\quad p \circ P_2 = T$

$q \circ P_2 = F$

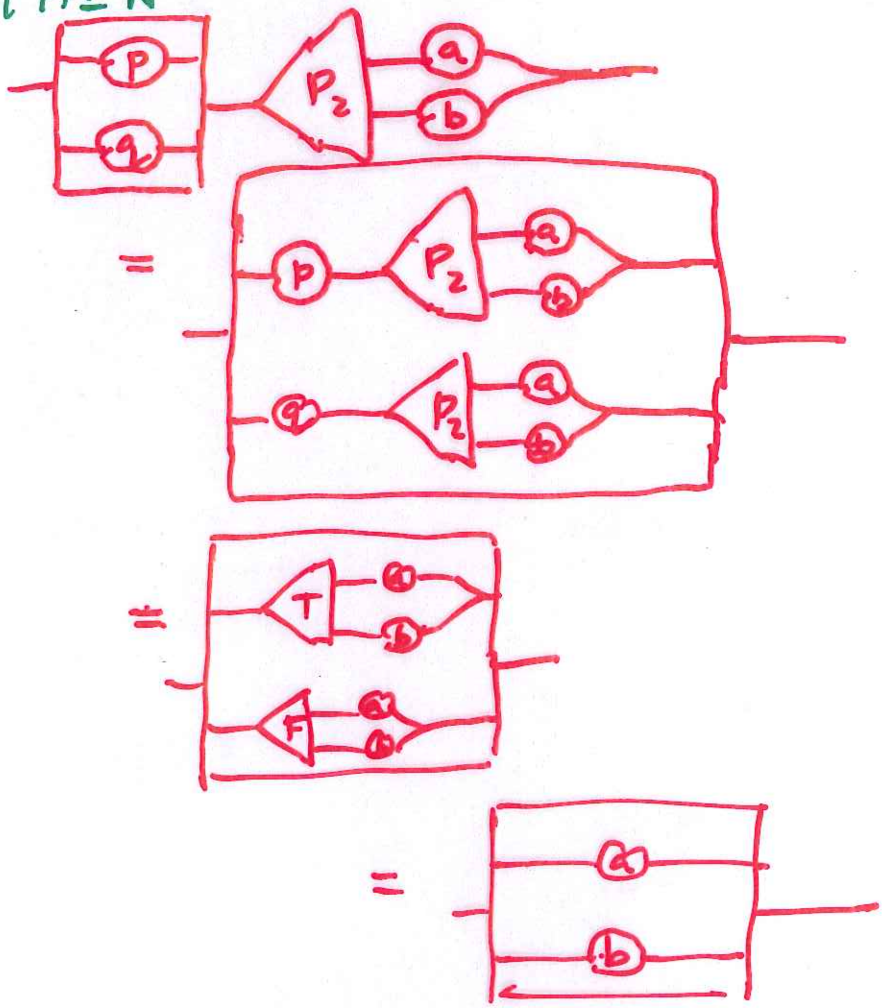
COMPUTATION

$\phi \mapsto d$



$\in A(2)$

THEN



DELICATE POINT

Start with $\mathcal{L}$

$$\mathcal{L} \longmapsto \mathcal{L}(0) \longmapsto \mathcal{U}_{\mathcal{L}(0)}$$

gives $\mathcal{L} \longrightarrow \mathcal{U}_{\mathcal{L}(0)}$ $\cong$ by category theory.

BUT

Start with A

$$A \longmapsto \mathcal{U}_A \longmapsto \mathcal{U}_A(0)$$

Then to show $\mathcal{U}_A(0) \cong A$ requires some care.

FUNDAMENTAL THEOREM

$\mathcal{L} \longmapsto \mathcal{L}(0)$ and $A \longmapsto \mathcal{U}_A$
give an equivalence

λ-theories $\simeq$ Λ-algebras

in such a way that every
λ-theory $\mathcal{L}$ is isomorphic
to the theory $\Lambda_{\mathcal{L}(0)}$ of
Λ-algebra extensions
of $\mathcal{L}(0)$