

Miraculous Cancellation and Pick's Theorem.

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ABSTRACT. We show that the Cappell–Shaneson version of Pick's theorem for simple lattice polytopes is a consequence of a general relation between characteristic numbers of virtual submanifolds dual to the characteristic classes of a stably almost complex manifold. This relation is analogous to the miraculous cancellation formula of Alvarez-Gaume and Witten, and is imposed by the action of the Landweber–Novikov algebra in the complex cobordism ring of a point.

Introduction

Studying gravitational anomalies Alvarez-Gaume and Witten [1] discovered a remarkable cancellation which they called the miraculous cancellation formula. The formula is a consequence of the following relation between Pontryagin characteristic numbers of 12-dimensional oriented manifolds:

$$(1) \quad L(M) = 8A(M, T) - 32\hat{A}(M),$$

where $L(M)$ is the L -genus (the signature) of M corresponding to the power series $L(x) = x/\tanh(x)$, $\hat{A}(M)$ is the \hat{A} -genus of M corresponding to the power series $\hat{A}(x) = x/2\sinh(x/2)$, and $A(M, T)$ is the twisted \hat{A} -genus defined by cohomology Kronecker product

$$\left\langle \prod_j A(x_j) \left(\sum_j e^{x_j} + e^{-x_j} \right), [M] \right\rangle$$

($\pm x_j$ are the formal Chern roots of $\mathbb{C} \otimes \tau(M)$ in ordinary cohomology and $[M] \in H^{\dim M}(M, \mathbb{Q})$ is the fundamental class).

The miraculous cancellation formula was generalized by Kefeng Liu [22] to higher dimensions with the use of modular invariance of certain elliptic operators on the loop space, it was strengthened further in [17].

Motivated by this development Buchstaber and Veselov suggested in [5] that a general formula exists which relates characteristic numbers of virtual submanifolds Poincaré dual to cobordism characteristic classes of a given manifold. In particular, they found that the signatures of manifolds Poincaré dual to the tangent Pontryagin

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classes with values in complex cobordism of an oriented manifold M^{4n} are related as

$$L(M^{4n}) + \sum_{k=1}^n (-1)^k L([P_k^U(\tau(M^{4n}))]) = 0 \pmod{2^{\alpha(n)}},$$

where $\alpha(n)$ is the number of ones in the binary expansion of the number n . The general formula which expresses all relations between Chern characteristic numbers of virtual submanifolds was obtained in [14].

Various applications of miraculous cancellation type formulae to the questions of divisibility of topological invariants were found in [13, 14, 17, 23]. In this paper we show that Cappell–Shaneson version [9] of Pick’s theorem is, in fact, also a consequence of a cancellation formula for the Chern characteristic numbers of virtual submanifolds.

Classical Pick’s theorem [28] states that the area of a convex polygon P with all vertexes in the standard two-dimensional unit lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ can be expressed in terms of the number of lattice points inside the polygon as

$$Area(P) = Int(P) + \frac{Bd(P)}{2} - 1,$$

where $Int(P)$ is the number of lattice points strictly inside P and $Bd(P)$ is the number of lattice points on the boundary of P . Numerous generalizations (see [4, 15, 18, 19, 25, 29] and references there) of this statement to higher dimensions are based on the theory of toric varieties [12]. Toric varieties are algebraic varieties naturally associated with a simple lattice polytope in \mathbb{R}^n . The twisted Todd genus of a toric variety is the number of lattice points in the corresponding lattice polytope. Lack of elementary description of the Todd class results in complexity of combinatorial formulae for the number of lattice points [25, 29].

One may look at the classical Pick’s theorem from a slightly different angle. We may argue that in Pick’s theorem lattice points should be counted with different weights depending on the dimension of faces they belong to. Then for a lattice n -gon P Pick’s theorem takes the form

$$Area(P) = Int(P) + \frac{Edg(P)}{2} + \frac{Vert(P)}{4} - \frac{4-n}{4},$$

where $Edg(P)$ is the number of lattice points inside each edge of P and $Vert(P) = n$ is the number of vertexes of P . Cappell and Shaneson observed that for a general simple lattice polytope $P \subset \mathbb{R}^n$ the sum

$$\#(\text{rel.int}(P) \cap \mathbb{Z}^n) + \sum_{k=1}^n \left(\frac{1}{2}\right)^k \sum_{\substack{F \subset P \\ \dim F = (n-k)}} \#(\text{rel.int}(F) \cap \mathbb{Z}^n)$$

can be expressed in terms of the polytope algebra over the ring of infinite order constant coefficient differential operators on \mathbb{R}^n . In this paper we prove that for any stably almost complex manifold M^{2n} with some fixed cohomology class $w \in H^2(M^{2n}, \mathbb{Z})$

$$\begin{aligned} \langle \exp(w) \text{Td}(M), [M] \rangle + \sum_{k=1}^n \left(-\frac{1}{2}\right)^k \langle \exp(w_k) \text{Td}([c_k(\tau(M))]), [c_k(\tau(M))] \rangle = \\ = \langle \exp(w) \prod_{i=1}^m \frac{x_i/2}{\tanh(x_i/2)}, [M] \rangle, \end{aligned}$$

where $[c_k(\tau(M))]$ are manifolds Poincare dual to the cobordism Chern characteristic classes of $\tau(M)$, Td is the Todd class, x_i are the tangent Chern roots of M and $w_k \in H^2([c_k(\tau(M))], \mathbb{Z})$ are the pullbacks of w under the canonical embeddings $[c_k(\tau(M))] \subset M \times \mathbb{R}^N$. In the case of toric varieties this formula displays a duality between cobordism invariants of a smooth toric variety and its canonical submanifolds defined by faces of the corresponding polytope. Twisted Todd genera of these submanifolds are exactly the numbers of lattice points inside the closures of the corresponding faces and the Pick's type theorem reads as

$$(2) \quad \sharp(P \cap \mathbb{Z}^n) + \sum_{k=1}^n \left(-\frac{1}{2}\right)^k \sum_{\substack{F \subset P \\ \dim F = (n-k)}} \sharp(F \cap \mathbb{Z}^n) = \int_{M_P} \exp(w_P) \prod_{j=1}^m \frac{x_j/2}{\tanh(x_j/2)},$$

where w_P is the Kahler class of a toric variety M_P corresponding to the simple lattice polytope P , and x_i , $i = 1, \dots, m$, are Chern classes of the canonical line bundles corresponding to the codimension one faces of P . The right hand side of (2) is a weighted analogue of the twisted signature of M_P .

In lower dimensions the twisted signature can be developed explicitly. We use it for evaluation of a Pick's type theorem for a three-dimensional example. As an intermediate step in order to compute the twisted signature we obtain a general expression for monomial Chern characteristic numbers of a smooth toric variety of any dimension in terms of primitive vectors which define codimension one faces of the polytope.

The paper is organized as follows. The first three sections give a brief account of the facts used for the proof of the miraculous cancellation type formula. All information in these sections is well know to experts and is given mainly to set up the notations. The cancellation formula itself is proved in Section 4. In Section 5 we show how the cancelation formula of Alvarez-Gaume and Witten can be deduced within the same cobordism technique. In Section 6 we state basic facts about toric manifolds and give an expression for the Gysin map in terms of the fixed point data. Finally, in Section 7 we derive Cappell–Shaneson type formula and evaluate Pick's theorem for some low-dimensional examples.

1. Bordism Ring of Manifolds with Line Bundles

Let us consider a set of all pairs consisting of a compact stably almost complex manifold without boundary and a complex line bundle over it. We introduce an equivalence relation on this set by means of a standard construction from cobordism theory. We say that two pairs (M_1^n, η_1) and (M_2^n, η_2) are equivalent if there is a compact stably almost complex manifold W of dimension $n+1$ with a complex line bundle η over it such that

$$\partial W = M_1^n \cup M_2^n, \quad \eta|_{M_1^n} \cong \eta_1, \quad \eta|_{M_2^n} \cong \eta_2,$$

and the restrictions of the almost complex structure in the normal bundle $\nu(W)$ of W on M_i^n , $i = 1, 2$, coincide with $\nu(M_1^n)$ and $\overline{[1]}_{\mathbb{C}} \oplus \nu(M_2^n)$ respectively ($[1]_{\mathbb{C}}$ is the trivial complex line bundle). Where it is necessary we write $(M, \nu(M), \eta)$ instead of (M, η) to underline the fixed (in the normal bundle) stably almost complex structure on M .

The set of pairs (M, η) factorized by the above equivalence relation, is a graded Abelian group with an addition given by a disjoint union of manifolds. The inverse

to $(M, \nu(M), \eta)$ is $(M, \overline{[1]}_{\mathbb{C}} \oplus \nu(M), \eta)$. Zero is the empty set. We refer to this group as Ω_*^{LU} .

We introduce a multiplication in Ω_*^{LU} by the formula given on representatives of bordism classes:

$$[(M_1, \eta_1)] \cdot [(M_2, \eta_2)] = [(M_1 \times M_2, \eta_1 \otimes \eta_2)].$$

Obviously, Ω_*^{LU} equipped with such an operation becomes a commutative ring with a unit $(pt, [1]_{\mathbb{C}})$. We can easily calculate the ring Ω_*^{LU} .

PROPOSITION 1. *The graded Abelian group Ω_*^{LU} is isomorphic to the complex bordism group of the infinite complex projective space:*

$$\Omega_*^{LU} \cong U_*(CP(\infty), \emptyset).$$

Therefore, the ring Ω_*^{LU} is the polynomial ring $\Omega_*^U[[t]]$, where Ω_*^U is the complex bordism ring (complex cobordism group of a point).

For any pair of topological spaces (X, A) with basepoints we define the corresponding Ω_*^{LU} -bordism group $LU_*(X, A)$ which consists of all triples (M, η, f) , where M is a compact stably almost complex manifold with boundary, η is a complex line bundle over M and f is a continuous map:

$$f : (M, \partial M) \rightarrow (X, A),$$

factorized by the bordism relation:

$$(M_1, \eta_1, f_1) \sim (M_2, \eta_2, f_2),$$

if and only if there is a triple (W, η, F) , such that

$$\partial W = M_1 \bigcup M_2 \bigcup V, \quad F : (W, V) \rightarrow (X, A),$$

$$F|_{M_i} = f_i, \quad \nu|_{M_1} = \nu_1, \quad \nu|_{M_2} = \overline{[1]}_{\mathbb{C}} \oplus \nu_1.$$

Denote the Thom space of the tautological complex vector bundle η_n over $BU(n)$ by $MU(n)$. Let E_{2k} , $k = 0, 1, \dots$, be the wedges $CP(\infty)_+ \wedge MU(k)$, where X_+ is a disjoint union of a topological space X and a point $\{pt\}$, we will also use an identification $X_+ = (X, \emptyset)$. Set $E_{2k+1} = \Sigma E_{2k}$. Define a sequence of maps:

$$f_k : \Sigma E_k \rightarrow E_{k+1}$$

by the formulae

$$f_{2k} = id,$$

$$f_{2k+1} : \Sigma E_{2k+1} = \Sigma^2 E_{2k} = CP(\infty)_+ \wedge \Sigma^2 MU(k) \xrightarrow{id \wedge j_k} CP(\infty)_+ \wedge MU(k+1),$$

where $j_k : \Sigma^2 MU(k) \rightarrow MU(k+1)$ is the standard embedding.

There is a sequence of canonical maps $i_n : S^n \rightarrow E_n$ which is inherited from the canonical embedding $S^{2n} \subset MU(n)$. We also define a graded paring for any two spaces E_k and E_m :

$$\mu_{km} : E_k \wedge E_m \rightarrow E_{k+m},$$

by the formula given on even spaces E_n :

$$\begin{aligned} \mu_{2k, 2m} : E_{2k} \wedge E_{2m} &= (CP(\infty) \times CP(\infty))_+ \wedge MU(k) \wedge MU(m) \xrightarrow{\otimes \wedge i} \\ &\rightarrow CP(\infty)_+ \wedge MU(k+m), \end{aligned}$$

where $\otimes : CP(\infty) \times CP(\infty) \rightarrow CP(\infty)$ is the classifying map for the tensor product of Hopf's line bundles over $CP(\infty)$, $i : MU(k) \wedge MU(m) \rightarrow MU(k+m)$ is the canonical embedding.

PROPOSITION 2. *The set of spaces $\{E_k\}_{k=0}^\infty$ and maps $\{f_k, j_l, \mu_{mn}\}$ forms a multiplicative spectrum, and, thus, defines a (co)homology theory:*

$$h_s(X, A) = \lim_{t \rightarrow \infty} \pi_{s+t}(X/A \wedge E_t),$$

$$h^s(X, A) = \lim_{t \rightarrow \infty} [\Sigma^t(X/A), E_{s+t}]$$

The corresponding cohomology theory is multiplicative.

By the standard Thom transversality arguments we deduce

PROPOSITION 3. *The bordism theory $LU_*(\cdot)$ coincides with the homology theory $h_*(\cdot)$.*

We denote the corresponding cobordism theory by $LU^*(\cdot) \cong h^*(\cdot)$.

2. Poincare Duality and the Euler Class

Let us consider a compact stably almost complex manifold M^n without boundary. Assume that this manifold is equipped with a complex line bundle ζ . Any element $\alpha \in LU^k(M^n)$ can be realised as a compact stably almost complex submanifold L , $\dim L = n - k$, in $M^n \times \mathbb{R}^N$ (N is large) without boundary, equipped with a complex line bundle ζ_L and with a fixed stable complex structure ν in the normal bundle of the embedding $L \subset M^n \times \mathbb{R}^N$ satisfying $\nu(L) \cong_{\mathbb{C}} \nu \oplus \nu(M \times \mathbb{R}^N)|_L$. Denote the embedding $L \subset M^n \times \mathbb{R}^N$ by i and the projection $M^n \times \mathbb{R}^N \rightarrow M^n$ by p . Define a map:

$$D : LU^*(M^n) \rightarrow LU_{n-*}(M^n)$$

by the formula

$$D[(L, \nu, \zeta_L)] = [(L, \bar{\zeta}_L \otimes i^* p^* \zeta)]$$

PROPOSITION 4. *For any compact stably almost complex manifold M^n equipped with a line bundle ζ the map*

$$D = D(M^n; \zeta) : LU^*(M^n) \rightarrow LU_{n-*}(M^n)$$

is an isomorphism.

COROLLARY 1. *For any compact closed stably almost complex manifold M^n equipped with a complex line bundle η , the fundamental class of M^n , i.e. D -dual to $1 \in LU^0(M^n)$, is*

$$[M^n, \eta] = [(M^n, id, \eta)] \in LU_n(M^n).$$

Let us consider a complex n -dimensional vector bundle ξ over a base space X . Assume that X is equipped with a complex line bundle ζ .

DEFINITION 1. *The canonical Thom class of (ξ, ζ) is an element $u(\xi, \zeta) \in LU^{2n}(T\xi)$ represented by the map*

$$\alpha : T\xi \xrightarrow{p \wedge id} X_+ \wedge T\xi \xrightarrow{f_{\bar{\zeta}} \wedge g_{\xi}} CP(\infty)_+ \wedge MU(n),$$

where $p(x, v_x) = x$ is the projection outside infinity, $f_{\bar{\zeta}} : X \rightarrow CP(\infty)$ is the map classifying $\bar{\zeta}$, $g_{\xi} : T\xi \rightarrow MU(n)$ is the map classifying ξ . The homotopy class of α lies in $[T\xi, CP(\infty)_+ \wedge MU(n)]$ and, thus, it defines an element

$$[\alpha] \in \lim_{k \rightarrow \infty} [\Sigma^{2k} T\xi, CP(\infty)_+ \wedge MU(n+k)] = LU^{2n}(T\xi);$$

PROPOSITION 5. For a pair (ξ, ζ) of two complex vector bundles over a finite CW-complex X ($\dim_{\mathbb{C}} \xi = n$, $\dim_{\mathbb{C}} \zeta = 1$) the multiplication by $u(\xi, \zeta)$

$$LU^*(X, \emptyset) \longrightarrow LU^{*+2n}(T\xi),$$

is an isomorphism.

Consider a complex n -dimensional vector bundle ξ over a topological space X . Assume that X is equipped with a complex line bundle ζ . Denote the classifying maps for ξ and $\bar{\zeta}$ by f, g respectively:

$$\begin{aligned} f : X &\rightarrow BU(n), & \xi &= f^*\eta_n, & \hat{f} : T\xi &\rightarrow MU(n); \\ g : X &\rightarrow CP(\infty), & \bar{\zeta} &= g^*\eta_1. \end{aligned}$$

Let $s_0 : X \rightarrow T\xi$ be the embedding onto zero section of ξ .

DEFINITION 2. The Euler class of the pair (ξ, ζ) is

$$\chi(\xi, \zeta) = s_0^*u(\xi, \zeta);$$

$$\chi(\xi, \zeta) = [g \wedge (\hat{f} \circ s_0)] \in \lim_{t \rightarrow \infty} [\Sigma^{2t}(X_+), CP(\infty)_+ \wedge MU(n+t)] = LU^{2n}(X, \emptyset).$$

Once we have defined the Euler class we can continue and introduce the other characteristic classes for (ξ, ζ) following techniques from [6, 10].

Let $\mathbb{C}G_k^n(\xi)$ be the complex grassmannization of the complex vector bundle ξ . Denote the canonical k -dimensional complex vector bundle over $\mathbb{C}G_k^n(\xi)$ by $\xi(k)$. Let $p_k : \mathbb{C}G_k^n(\xi) \rightarrow X$ be the projection on the base space X , and $\tau(p_k) : S^N \wedge X_+ \rightarrow S^N \wedge \mathbb{C}G_k^n(\xi)_+$ be the transfer map of Becker–Gottlieb [3] for fibre bundle $(\mathbb{C}G_k^n(\xi), \mathbb{C}G_k^n, X, p_k)$.

DEFINITION 3. Characteristic classes $c_{km}^{LU}(\xi, \zeta)$ with values in $LU^*(X, \emptyset)$ of (ξ, ζ) are defined by

$$c_{km}^{LU}(\xi, \zeta) = \tau(p_k)^* \{ \chi(\xi(k), p_k^*\zeta^m) \} \in LU^{2k}(X, \emptyset),$$

here $\dim_{\mathbb{C}} \xi = n$, $\dim_{\mathbb{C}} \zeta = 1$.

3. The Chern-Dold character

For any multiplicative cohomology theory $h^*(\cdot)$ there is a unique multiplicative transformation

$$ch_h : h^*(\cdot) \rightarrow H^*(\cdot; h^*(pt) \otimes \mathbb{Q}),$$

such that for a point it is the identical embedding. Tensoring with the rational numbers this transformation is an isomorphism of cohomology theories.

Consider a compact stably almost complex manifold M^{2n} with a fixed complex structure in its normal bundle ν , $\dim_{\mathbb{C}} \nu = m$. Let ζ be a complex line bundle over M . Chose $1 \in LU^0(M^{2n}, \emptyset)$. This element can be represented by a triple $(M^{2n}, [1]_{\mathbb{C}}, id)$.

PROPOSITION 6. Let $p : M^{2n} \rightarrow \{pt\}$ be the constant map. In the notations described above:

$$[M^{2n}, \zeta] = ch_{LU} \circ p_!^{LU}(1) = \langle T_{LU}(\nu), [M^{2n}] \rangle,$$

where $p_!^{LU} : LU^*(M^{2n}, \emptyset) \rightarrow LU^{*-2n}(S^0)$ is the Gysin map, T_{LU} is the Todd class of the transformation ch_{LU} , $[M^{2n}]$ is the fundamental class of the manifold M^{2n} in homology $H_*(M^{2n}; \Omega_*^{LU} \otimes \mathbb{Q})$.

Proof. The first equality follows from Corollary 1 and the second equality is the standard consequence of the Riemann–Roch theorem.

For any compact closed stably almost complex manifold M^{2n} equipped with a complex line bundle ζ we define normal mixed characteristic numbers:

$$c_I(M, \zeta) = \langle c_I(\nu(M))c_1^{n-|I|}(\zeta); [M^{2n}] \rangle,$$

where c_I is the Chern class in ordinary cohomology corresponding to the partition $I = (i_1, \dots, i_k)$, $|I| = i_1 + \dots + i_k \leq n$. It is a well-known fact [31] that two compact closed stably almost complex manifolds M_1, M_2 equipped with complex line bundles ζ_1, ζ_2 respectively represent the same element in Ω_*^{LU} if and only if they have the same mixed characteristic numbers.

Let $S(x)$ be an infinite power series in x defined by the formula:

$$S(x) = \prod_{i=1}^{\infty} (1 - y_i x)^{-1},$$

where $y_i, i = 1, \dots$, are independent variables. Define a map:

$$S_t : \Omega_*^{LU} \rightarrow \Lambda_y[t],$$

$$S_t([(M^{2n}, \nu(M^{2n}), \zeta)]) = \langle e^{-tc_1(\zeta)} \prod_j S(x_j), [M^{2n}] \rangle$$

where Λ_y is the ring of symmetric polynomials over \mathbb{Q} in variables $y_i, i = 1, 2, \dots$, and $x_j, j = 1, 2, \dots$, are the Chern roots of the normal bundle $\nu(M^{2n})$.

PROPOSITION 7. *The map S_t is a monomorphism which induces an isomorphism*

$$S_t : \Omega_*^{LU} \otimes \mathbb{Q} \rightarrow \Lambda_y[t].$$

COROLLARY 2. *For every ring homomorphism $\phi : \Omega_*^{LU} \rightarrow \mathbb{Q}$ there is such a rational valued specialization for elementary symmetric functions $e_i = e_i(y_1, y_2, \dots)$ and for variable t , that S_t for this specialization coincides with ϕ .*

Let S_t^* be the transformation

$$H^*(\cdot; \Omega_*^{LU} \otimes \mathbb{Q}) \rightarrow H^*(\cdot; \Lambda_y[t]),$$

induced by the coefficient ring homomorphism S_t .

PROPOSITION 8. *The Todd class of the composition $\tau = S_t^* \circ ch_{LU}$ is*

$$T_\tau(x, z) = e^{-tz} \prod_{i=1}^{\infty} (1 - y_i x)^{-1}.$$

In this expression for a pair (ξ, ζ) of complex bundles over the base space X ($\dim_{\mathbb{C}} \zeta = 1$) the variable x is fixed for the Chern roots in ordinary cohomology of ξ , and the variable z is fixed for the first Chern class of ζ .

Let us calculate $LU^*(CP(n), \emptyset)$. We fix the canonical line bundle η_1 over $CP(n)$. Observe also that $CP(n+1) \approx T\bar{\eta}_1$. Thus we have

LEMMA 1. *There is an isomorphism*

$$LU^{*+2}(CP(n+1)) \cong LU^*(CP(n), \emptyset),$$

via multiplication by the Thom class $u(\bar{\eta}_1, \eta_1)$.

COROLLARY 3. *The ring $LU^*(CP(n), \emptyset)$ is a polynomial ring $\Omega_*^{LU}[u]/\{u^{n+1} = 0\}$. It is additively generated by $[(CP(k), \eta_1^{n-k}, i_k)]$, $i_k : CP(k) \xrightarrow{\nu_k} CP(n)$, $k = 1, \dots, n$. The multiplication is given by the rule*

$$[(CP(k), \eta_1^{n-k}, i_k)] \cdot [(CP(m), \eta_1^{n-m}, i_m)] = [(CP(k+m-n), \eta_1^{2n-k-m}, i_{k+m-n})],$$

PROPOSITION 9. *Via the map*

$$\tau = S_t^* \circ ch_{LU} : LU^*(CP(\infty), \emptyset) \rightarrow H^*(CP(\infty); \Lambda_y[t])$$

the canonical generator $u \in LU^(CP(\infty), \emptyset)$ goes to*

$$xe^{-tx} \prod_{i=1}^{\infty} (1 - y_i x)^{-1}$$

where x is the canonical generator of $H^(CP(\infty), \Lambda_y[t])$.*

4. General Cancellation Formula

The theory LU^* possesses an algebra of stable operations similar to that of Landweber–Novikov algebra in complex cobordism [20, 26]. Namely, we can combine them in one multiplicative operation which on the Thom class $u(\eta_n, \eta_1)$ of $(CP(\infty)_+)^{\wedge} MU(n)$ takes the following value:

$$\mathfrak{S}_{(z)}(u(\eta_n, \eta_1)) = u(\eta_n, \eta_1) \left(1 + \sum_{\lambda > 0} h_{\lambda}(z) m_{\lambda}(\eta_n) \right),$$

where $h_{\lambda}(z)$ is the complete symmetric function corresponding to the partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq 0)$ [24], $m_{\lambda}(\eta_n)$ is the LU^* -Chern class corresponding to the monomial symmetric function m_{λ} and a trivial line bundle. Similar to [14] we conclude that

PROPOSITION 10. *The composition $\Theta = S_{(y,t)}^* \circ ch_{LU} \circ \mathfrak{S}_{(z)}$:*

$$\Theta : LU^*(\cdot) \rightarrow H^*(\cdot, \Lambda_y \otimes \Lambda_z \otimes \mathbb{Q}[t])$$

is a multiplicative transformation of cohomology theories and its Todd class is

$$T_{\Theta}(\eta, \zeta) = e^{-tu} \prod_{i=1}^n \left(\prod_{j=1}^{\infty} (1 - y_j x_i)^{-1} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z_k x_i}{\prod_{l=1}^{\infty} (1 - y_l x_i)} \right)^{-1} \right),$$

where u is the first Chern class of the complex line bundle ζ , x_i , $i = 1, \dots, n$, are the Chern roots in integral cohomology of the complex vector bundle η , $n = \dim_{\mathbb{C}} \eta$.

Applying the Riemann–Roch theorem to the transformation Θ and to the constant map $p : M \rightarrow \{pt\}$ with LU -oriented manifold (M^{2n}, ζ) , we deduce the following theorem:

THEOREM 1. *Let $([m_{\lambda}(\nu(M^{2n}))], i_{\lambda}^* \zeta)$ be manifolds LU -dual to the LU -Chern classes of the normal bundle $\nu(M^{2n})$ which correspond to the monomial symmetric functions m_{λ} , $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq 0)$. Then*

$$\begin{aligned} & \langle T_{\Theta}(\nu(M^{2n}), \zeta), [M^{2n}] \rangle = \\ & = \langle T_{\tau}(\nu(M^{2n}), \zeta), [M^{2n}] \rangle + \sum_{\lambda > 0} h_{\lambda}(z) \langle T_{\tau}(\nu_{\lambda}, i_{\lambda}^* \zeta), [m_{\lambda}(\nu(M^{2n}))] \rangle, \end{aligned}$$

where ν_{λ} is the normal bundle of $m_{\lambda}(\nu(M^{2n}))$ with the induced stable almost complex structure.

Let M be a stably almost complex manifold and $x_i, i = 1, 2, \dots$, be the Chern roots of the stable complex tangent bundle to M . We define the Todd class of M by

$$\mathrm{Td}(M) = \prod_{i=1}^{\infty} \frac{x_i}{1 - e^{-x_i}}.$$

This is a symmetric polynomial in $x_i, i = 1, 2, \dots$, which can be expressed as a series in tangent Chern classes of M in ordinary cohomology.

THEOREM 2. *Let M be a stably complex manifold and $w \in H^2(M, \mathbb{Z})$. Then for the manifolds $[c_k(\tau(M))]$ dual to the complex cobordism Chern classes c_k^U of $\tau(M)$ the following formula holds*

$$\begin{aligned} & \langle \exp(w) \prod_{i=1}^m \frac{x_i/2}{\tanh(x_i/2)}, [M] \rangle = \\ & = \langle \exp(w) \mathrm{Td}(M), [M] \rangle + \sum_{k=1}^n \left(-\frac{1}{2}\right)^k \langle \exp(w_k) \mathrm{Td}([c_k(\tau(M))]), [c_k(\tau(M))] \rangle, \end{aligned}$$

where x_i are the tangent Chern roots of M and $w_k \in H^2([c_k(\tau(M))], \mathbb{Z})$ are the pullbacks of w under the canonical embeddings $[c_k(\tau(M))] \subset M \times \mathbb{R}^N$.

Proof. Let us substitute the following values for the variables t, z_i and y_j in the series $T_{\Theta}(\eta, \zeta)$, where η is the normal bundle of M , and ζ is a line bundle over M with $c_1(\zeta) = w$:

$$t = -1, \quad z_1 = 1/2, \quad z_2 = z_3 = \dots = 0,$$

$$\prod_{j=1}^{\infty} (1 - y_j x)^{-1} = \frac{1 - e^{-x}}{x}.$$

For these values of variables we obtain

$$T_{\Theta}(\eta, \zeta) = e^u \prod_{i=1}^n \frac{1 - e^{-x_i}}{x_i} \frac{2}{1 + e^{-x_i}} = e^u \prod_{i=1}^n \frac{\tanh(x_i/2)}{x_i/2},$$

where x_i are the Chern roots of η and $u = c_1(\zeta)$. Therefore, for the tangent Chern roots $x'_i, i = 1, 2, \dots$, the left hand side of the formula in Theorem 1 becomes

$$\langle \exp(w) \prod_{i=1}^m \frac{x'_i/2}{\tanh(x'_i/2)}, [M] \rangle.$$

Because $c_k(\tau(M)) = (-1)^k h_k(\nu(M))$ for our choice of variables z_i , the right hand side of Theorem 1 is exactly

$$\langle \exp(w) \mathrm{Td}(M), [M] \rangle + \sum_{k=1}^n \left(-\frac{1}{2}\right)^k \langle \exp(w_k) \mathrm{Td}([c_k(\tau(M))]), [c_k(\tau(M))] \rangle.$$

Therefore, the statement of the theorem is verified.

5. Cancellation Formula of Alvarez-Gaume and Witten

The gravitational anomaly cancellation formula (1) of Alvarez-Gaume and Witten [1] can also be obtained from Theorem 1 by a suitable substitution. Let us consider a 12-dimensional compact stably almost complex manifold M without boundary. Choose values of variables z_i , $i = 1, 2, \dots$, in such a way that

$$\prod_{k=1}^{\infty} (1 - z_k x) = \sqrt{1 + \frac{1}{4}x^2},$$

we also define variables y_j , $j = 1, 2, \dots$, to satisfy

$$\prod_{j=1}^{\infty} (1 - y_j x)^{-1} = \frac{\sinh(x/2)}{x/2}.$$

Using Theorem 1 with this choice of z_i , y_j , $i, j = 1, 2, \dots$, and $t = 0$, we obtain:

$$(3) \quad \frac{1}{2^6} L(M) = \hat{A}(M) + \frac{1}{2^3} \hat{A}([P_1(\tau(M))]) + \\ + \frac{1}{2^7} \hat{A}(4[P_2(\tau(M))] - [P_1^2(\tau(M))]) + \frac{1}{2^{10}} (p_{111} - 4p_{12} + 8p_3),$$

where $[P_1(\tau(M))]$, $[P_2(\tau(M))]$, $[P_1^2(\tau(M))]$ are manifolds Poincaré dual to the corresponding tangent Pontryagin classes in cobordism, and p_{111} , p_{12} , p_3 are the tangent Pontryagin numbers.

From the same substitution it is easy to see that

$$\hat{A}([P_1(\tau(M))]) = \left\langle \frac{\partial}{\partial t} \prod_{i=1}^6 \frac{x_i \sqrt{1 + t(e^{x_i} + e^{-x_i} - 2)}}{\sinh(x_i/2)} \Big|_{t=0}, [M] \right\rangle = \\ = A(M, T) - 12\hat{A}(M),$$

where x_i^2 , $i = 1, \dots, 6$, are the tangent Pontryagin roots of the 12-dimensional stably almost complex manifold M , and $A(M, T)$ is the twisted \hat{A} -genus defined by

$$\left\langle \prod_j A(x_j) \left(\sum_j e^{x_j} + e^{-x_j} \right), [M] \right\rangle.$$

Observe that if W is a 4-dimensional stably almost complex manifold then $\hat{A}(W) = \mathbf{s}_2([W])/24$, where \mathbf{s}_2 is the second Landweber-Novikov operation acting on $\Omega_U^*(\{pt\}, \emptyset)$, and $[W] \in \Omega_U^4(\{pt\}, \emptyset)$ is the cobordism class defined by W . Using standard rules of the Landweber-Novikov algebra, we deduce that

$$\mathbf{s}_2(4[P_2(\tau(M))] - [P_1^2(\tau(M))]) = 12p_{12} - 24p_3 - 3p_{111}.$$

Collecting similar terms in (3) we arrive at Alvarez-Gaume and Witten formula.

COROLLARY 4. *For a 12-dimensional compact stably almost complex manifold without boundary*

$$L(M) = 8A(M, TM) - 32\hat{A}(M).$$

REMARK 1. *This formula is true for any compact oriented 12-dimensional manifold without boundary, as it only depends on Pontryagin classes.*

6. Toric Manifolds

A toric variety is a closure of an orbit of an algebraic action of T^n , and the quotient of the variety with respect to this action is a simple polytope $P \subset \mathbb{R}^n$ (recall that a polytope is simple if at every its vertex exactly n faces of codimension one meet each other). Toric varieties are not always smooth. Polytopes for which the corresponding toric variety is smooth, are regular (or Delzant) polytopes and varieties themselves in this case are toric manifolds. More precisely, a simple lattice polytope is regular if the edges emanating from each vertex lie along vectors that generate the lattice \mathbb{Z}^n . Each face F_i , $i = 1, \dots, m$, of codimension one of a regular polytope P corresponds to a submanifold M_{F_i} of codimension two in the toric manifold M_P , $\dim M_P = 2n$. The torus action $T^n \times M_P \rightarrow M_P$ has a one-dimensional subgroup $S^{F_i} \subset T^n$ which keeps M_{F_i} fixed. S^{F_i} can be described by a primitive vector $\lambda_i = (\lambda_i^1, \dots, \lambda_i^n)$ of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ which is inward normal to the face $F_i \subset P \subset \mathbb{R}^n$. At each vertex $p \in P$ we get n vectors λ_{i_j} , $j = 1, \dots, n$, corresponding to each of the faces in the intersection:

$$p = F_{i_1} \cap \dots \cap F_{i_n}.$$

Together, λ_{i_j} , $j = 1, \dots, n$, form a basis of $\mathbb{Z}^n \subset \mathbb{R}^n$.

There is also a smooth manifold \mathfrak{L}_P of dimension $n+m$ where m is the number of codimension one faces of P such that M^{2n} is a quotient of \mathfrak{L}_P over the action of a torus T^{m-n} . The manifold \mathfrak{L}_P can be described as follows. Let us list all codimension one faces of P :

$$\mathfrak{F} = \{F_1, F_2, \dots, F_m\}.$$

For every face F_i in \mathfrak{F} we define the corresponding one-dimensional coordinate subgroup of $T^{\mathfrak{F}} = T^m$ by T^{F_i} . Now for a general face G of P let

$$T^G = \prod_{G \subset F_i} T^{F_i} \subset T^{\mathfrak{F}}.$$

DEFINITION 4. *For a given simple polytope we define*

$$\mathfrak{L}_P = (T^{\mathfrak{F}} \times P^n) / \sim,$$

where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in T^{G(q)}$.

(In this definition $G(q)$ is the face of P which contains q strictly in its interior.)

For a toric manifold M_P we define the characteristic homomorphism $\lambda : T^{\mathfrak{F}} \rightarrow T^n$, by means of isomorphisms

$$\lambda(T^{F_i}) \stackrel{\lambda_i}{\cong} S^{F_i}.$$

This is a characteristic map in the sense of [7, 11]. The kernel $H(\lambda)$ of λ is $(m-n)$ -dimensional subgroup of $T^{\mathfrak{F}}$ and as such it acts on \mathfrak{L}_P . This action is free and the factor space

$$\mathfrak{L}_P \xrightarrow{H(\lambda)} \mathfrak{L}_P / H(\lambda) \cong M_P$$

is the corresponding toric manifold. For each co-dimension one face F_i of the polytope P we define a representation:

$$\rho_i : T^{\mathfrak{F}} \rightarrow GL(1, \mathbb{C}) \cong \mathbb{C}^*$$

by means of a projection $T^{\mathfrak{F}} \rightarrow T^{F_i} \subset GL(1, \mathbb{C})$. Consider a one-dimensional complex line bundle L_i over $BP = ET^{\mathfrak{F}} \times_{T^{\mathfrak{F}}} \mathfrak{L}_P$ given by

$$L_i = ET^m \times_{(T^{\mathfrak{F}}, \rho_i)} (\mathfrak{L}_P \times \mathbb{C}).$$

The following theorem [11] describes the equivariant cohomology of \mathfrak{L}_P .

THEOREM 3. *The space $ET^m \times_{T^{\mathfrak{F}}} \mathfrak{L}_P$ is homotopy equivalent to $ET^n \times_{T^n} M_P^{2n}$. The cohomology ring*

$$H^*(ET^m \times_{T^{\mathfrak{F}}} \mathfrak{L}_P, \mathbb{Z}) \cong H^*(ET^n \times_{T^n} M_P^{2n}, \mathbb{Z})$$

is isomorphic to the face ring of P

$$\mathbb{Z}[v_1, \dots, v_m]/I, \quad v_i = c_1(L_i).$$

Under the projection $\pi_P : ET^n \times_{T^n} M_P^{2n} \rightarrow BT^n$ the generator

$$u_j \in H^*(BT^n, \mathbb{Z}) \cong \mathbb{Z}[u_1, \dots, u_n]$$

goes into

$$\pi_P^*(u_j) = \lambda_1^j v_1 + \dots + \lambda_m^j v_m \in H^*(ET^n \times_{T^n} M_P^{2n}, \mathbb{Z}).$$

At each vertex $p = F_{i_1} \cap \dots \cap F_{i_n}$ of P we get n linear independent vectors $\lambda_{i_1}, \dots, \lambda_{i_n}$ of \mathbb{Z}^n . Let $M_p^t = (\mu_{p, i_1}, \dots, \mu_{p, i_n})$ be an $n \times n$ column-matrix $(\mu_{p, j} = (\mu_{p, j}^1, \dots, \mu_{p, j}^n)^t)$ such that

$$M_p \cdot \Lambda_p = I_n, \quad \Lambda_p = (\lambda_{i_1}, \dots, \lambda_{i_n}),$$

and I_n is the $n \times n$ -identity matrix. Pick a generic vector $\vec{u} \in \mathbb{Z}^n$ which is not orthogonal to any of $\mu_{p, j}$.

LEMMA 2. *Under the Gysin map*

$$(\pi_P)_! : H^*(ET^n \times_{T^n} M_P^{2n}, \mathbb{Z}) \rightarrow H^*(BT^n, \mathbb{Z})$$

the n th power of the first Chern class of L_i goes into

$$(4) \quad \sum_{\substack{i_1 < \dots < i_{n-1} \neq i \\ p_I = F_i \cap F_{i_1} \cap \dots \cap F_{i_{n-1}} \neq \emptyset}} \frac{\langle \mu_{p_I, i}, \vec{u} \rangle^{n-1}}{\prod_{j=1}^{n-1} \langle \mu_{p_I, i_j}, \vec{u} \rangle},$$

which is independent of the choice of a generic vector \vec{u} .

Proof. This is an elementary consequence of the fixed point theory for compact Lie group actions [2, 27, 30]. We write down the fixed point formula for the action of T^n on M_P . At the fixed point corresponding to the vertex p_I we get a simple expression in terms of $v_i, v_{i_1}, \dots, v_{i_{n-1}}$:

$$\frac{v_i^n}{v_i \cdot v_{i_1} \cdots v_{i_{n-1}}}.$$

Rewriting it uniformly in terms of $u_1, \dots, u_n \in H^*(BT^n, \mathbb{Z})$ and substituting $u_j = \vec{u}_j$ leads to the expression in the statement.

Let $\omega = (w_1 \geq w_2 \geq \dots \geq 0)$ be a partition of n of length l . Denote by $s_\omega(\eta)$ the Chern class of η corresponding to the monomial symmetric function m_ω . Similar to Lemma 2 (see also [8]) we deduce

THEOREM 4. *For a smooth toric variety M^{2n} its Chern number $s_\omega(M)$ corresponding to a partition ω of n can be computed in terms of the fixed point data by means of a formula:*

$$s_\omega(M) = \sum_{\substack{I_1=\{i_1<\dots<i_l\} \\ I_2=\{i_{l+1}<\dots<i_n\}}} \sum_{\sigma \in S_l(\omega)} \frac{\prod_{j=1}^l \left(\mu_{p_I, i_j}^1 u_1 + \dots + \mu_{p_I, i_j}^n u_n \right)^{w_{\sigma(j)} - 1}}{\prod_{j=l+1}^n \left(\mu_{p_I, i_j}^1 u_1 + \dots + \mu_{p_I, i_j}^n u_n \right)}$$

where the first summation is taken over $I_1 \cap I_2 = \emptyset$ and with

$$p_I = F_{i_1} \cap \dots \cap F_{i_l} \cap F_{i_{l+1}} \cap \dots \cap F_{i_n} \neq \emptyset.$$

The formula is independent of the choice of a generic vector $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$.

In dimension three all formulae can be rewritten just in terms of λ_j because

$$\mu_{p, i} = \pm \frac{[\lambda_{p, j}, \lambda_{p, k}]}{\det \Lambda_p}.$$

In particular,

$$(5) \quad (\pi_P)_! (c_1(L_i)^3) = \sum_{\substack{i_1 < i_2 \neq i \\ p_I = F_{i_1} \cap F_{i_2} \neq \emptyset}} \frac{\langle \lambda_{p_I, i_1}, \lambda_{p_I, i_2}, \vec{u} \rangle^2}{\langle \lambda_{p_I, i_1}, \lambda_{p_I, i_1}, \vec{u} \rangle \langle \lambda_{p_I, i_2}, \lambda_{p_I, i_2}, \vec{u} \rangle}$$

where for three vectors $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ we denote the corresponding oriented volume by $\langle \vec{x}, \vec{y}, \vec{z} \rangle$.

7. Pick's Theorem

In this section we apply Theorem 1 for the case of toric manifolds and its Chern classes. Recall that the m -th tangent Chern class $c_m(\tau(M))$ of a toric manifold $M = M_P$ associated to a simple polytope P of dimension n is given by a dual cohomology class to the sum of fundamental classes of toric submanifolds M_F associated to $(n-m)$ -dimensional faces $F = F_j^{(n-m)}$ of P . Let w_P be the Kahler form on M_P . It is well known [16] that the equivariant class of the canonical Kahler form w_P on M_P is

$$-a_1 v_1 - \dots - a_m v_m \in H^2(ET^n \times_{T^n} M_P^{2n}, \mathbb{Z}),$$

where a_j , $j = 1, \dots, m$, are defined by means of a system of inequalities:

$$x \in P \iff \langle x, \lambda_i \rangle \geq a_i, \quad i = 1, \dots, m.$$

We also set $x_j = i^*(v_j)$ where

$$i : M^{2n} \subset ET^n \times_{T^n} M^{2n} \approx ET^m \times_{T^m} \mathfrak{L}_P$$

is an embedding on any fibre.

COROLLARY 5. *The sum over all faces of a regular lattice polytope $P \subset \mathbb{R}^n$ of the numbers of integral points inside the closures of the faces taken with weights $(-1/2)^{n-\dim F}$, is the twisted signature of the corresponding toric variety M_P :*

$$(6) \quad \langle \exp(w_P) \prod_{i=1}^m \frac{x_i/2}{\tanh(x_i/2)}, [M_P] \rangle = \#(P \cap \mathbb{Z}^n) + \sum_{k=1}^n \left(-\frac{1}{2}\right)^k \sum_{\substack{F \subset P \\ \dim F = (n-k)}} \#(F \cap \mathbb{Z}^n),$$

where x_i are the tangent Chern roots of M_P (m is the number of codimension one faces).

Proof. This is a straightforward consequence of Theorem 2 as due to [12] the value

$$\langle \exp(w_F) \text{Td}(M_F), [M_F] \rangle$$

is exactly the number of integral points in the face F .

REMARK 2. Formula (6) from Corollary 5 is valid for any simple lattice polytope. In order to verify this one needs to apply an equivariant generalization of Theorem 2 to the fibre bundle $ET^m \times_{T^m} \mathfrak{L}_P \rightarrow BT^m$.

EXAMPLE 1. Let $n = 2$ and P be a convex polygon in \mathbb{R}^2 with m integral vertexes. Then the left hand side of (6) is:

$$\begin{aligned} \langle \exp(w_P) \prod_{i=1}^m \frac{x_i/2}{\tanh(x_i/2)}, [M_P] \rangle &= \langle \frac{w_P^2}{2}, [M_P] \rangle + \langle \prod_{i=1}^m \frac{x_i/2}{\tanh(x_i/2)}, [M_P] \rangle = \\ &= \text{Area}(P) + \frac{\sigma(M_P)}{4}, \end{aligned}$$

where $\sigma(M_P)$ is the signature of the corresponding toric manifold. For any toric variety

$$\sigma(M_P) = h_0(P) - h_1(P) + \cdots + (-1)^n h_n(P) = (-1)^n h_P(-1),$$

where (h_0, h_1, \dots, h_n) is the h -vector of P and

$$h_P(t) = h_0 t^n + \cdots + h_n$$

is the h -polynomial [21]. In particular, for $n = 2$

$$\sigma(M_P) = h_0(P) - h_1(P) + h_2(P) = 4 - m.$$

Combining these observations with Corollary 5 we get

$$(7) \quad \text{Area}(P) + \frac{4-m}{4} = (\text{Int} + \text{Bd}) - \frac{\text{Bd} + m}{2} + \frac{m}{4} = \text{Int} + \frac{\text{Bd}}{2} - \frac{m}{4},$$

where Int is the number of integral points inside polygon P and Bd is the number of integral points on the boundary of P . It is easy to see that (7) is equivalent to the classical Pick's theorem:

$$\text{Area}(P) = \text{Int} + \frac{\text{Bd}}{2} - 1.$$

EXAMPLE 2. Let $n = 3$ and P be a regular (Delzant) tetrahedron in \mathbb{R}^3 with 2-dimensional faces F_1, F_2, F_3, F_4 . Then the left hand side of (6) is

$$\begin{aligned} \langle \exp(w_P) \prod_{i=1}^4 \frac{x_i/2}{\tanh(x_i/2)}, [M_P] \rangle &= \langle \frac{w_P^3}{6}, [M_P] \rangle + \langle w_P \prod_{i=1}^4 \frac{x_i/2}{\tanh(x_i/2)}, [M_P] \rangle = \\ &= \text{Vol}(P) - \sum_{j=1}^4 a_j \langle \frac{x_j}{3} \sum_{i=1}^4 \frac{x_i^2}{4}, [M_P] \rangle = \\ &= \text{Vol}(P) - \sum_{j=1}^4 \frac{a_j}{4} \langle \sum_{i \neq j} \frac{x_i^2}{3}, [M_{F_j}] \rangle - \sum_{j=1}^4 a_j \langle \frac{x_j^3}{12}, [M_P] \rangle = \\ &= \text{Vol}(P) - \sum_{j=1}^4 \frac{a_j}{4} h_{F_j}(-1) - \sum_{j=1}^4 a_j \langle \frac{x_j^3}{12}, [M_P] \rangle. \end{aligned}$$

Using (5) we obtain for each fixed $j = 1, \dots, 4$:

$$\langle x_j^3, [M_P] \rangle = \sum_{\substack{i_1, i_2 \neq j \\ p = F_j \cap F_{i_1} \cap F_{i_2} \neq \emptyset}} \frac{\langle \lambda_{p_I, i_1}, \lambda_{p_I, i_2}, \vec{u} \rangle^2}{\langle \lambda_{p_I, j}, \lambda_{p_I, i_1}, \vec{u} \rangle \langle \lambda_{p_I, i_2}, \lambda_{p_I, j}, \vec{u} \rangle},$$

where $\vec{u} \in \mathbb{Z}^3 \in \mathbb{R}^3$ is an arbitrary vector. For some fixed $i_1, i_2 \neq j$ chose $\vec{u} = \lambda_{i_3}$, $i_3 \neq i_1, i_2, j$, because $\lambda_\alpha, \lambda_\beta, \lambda_\gamma$ form a basis of \mathbb{Z}^3 for any choice of pairwise different indices α, β, γ we obtain that

$$\langle x_j^3, [M_P] \rangle = \frac{\langle \lambda_{p_I, i_1}, \lambda_{p_I, i_2}, \lambda_{p_I, i_3} \rangle^2}{\langle \lambda_{p_I, j}, \lambda_{p_I, i_1}, \lambda_{p_I, i_3} \rangle \langle \lambda_{p_I, i_2}, \lambda_{p_I, j}, \lambda_{p_I, i_3} \rangle} = 1.$$

Thus,

$$\langle \exp(w_P) \prod_{i=1}^4 \frac{x_i/2}{\tanh(x_i/2)}, [M_P] \rangle = \text{Vol}(P) - \sum_{j=1}^4 \frac{a_j}{4} - \sum_{j=1}^4 \frac{a_j}{12}.$$

Let Int , Fac , Edg be the numbers of integral points inside polyhedron $P \subset \mathbb{R}^3$, all faces of P and all edges of P respectively. If Vert is the number of vertexes of P then the right hand side of (6) becomes

$$\begin{aligned} \text{Int} + \text{Fac} + \text{Edg} + \text{Vert} - \frac{\text{Fac} + 2\text{Edg} + 3\text{Vert}}{2} + \frac{\text{Edg} + 3\text{Vert}}{4} - \frac{\text{Vert}}{8} = \\ = \text{Int} + \frac{\text{Fac}}{2} + \frac{\text{Edg}}{4} + \frac{\text{Vert}}{8}. \end{aligned}$$

COROLLARY 6. For a regular (Delzant) tetrahedron in \mathbb{R}^3 :

$$\text{Int} + \frac{\text{Fac}}{2} + \frac{\text{Edg}}{4} + \frac{\text{Vert}}{8} = \text{Vol}(P) - \sum_{j=1}^4 \frac{a_j}{3}.$$

8. Conclusion

Applying the new cancellation formula for the twisted signature and Todd genera of smooth toric varieties we derived a Pick's type theorem for simple lattice polytopes. Explicit calculations by means of this formula are possible only in low-dimensional examples due to the lack of a simple combinatorial description for the L -class of toric varieties. Finding such a description could lead to an interesting new invariant of simple polytopes.

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