

Isometries

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1 Introduction

One of the beauties of mathematics is that though mathematics consists of very abstract notions and follows very strict rules, it is an experimental science and it does describe various phenomena of nature in a very successful way. Plane geometry nicely illustrates this experimental nature of mathematics. It came to life many thousands years ago as a practical tool for solving elementary planning problems in agriculture and irrigation, building construction, travel planning and even in military actions. The abstract way of thinking turned out to be so successful that ancient Greeks made it into a divine subject. That time probably was the birth of mathematics. For many years two-dimensional geometry was the forefront of contemporary mathematics, and even after the so-called "devil of abstract" algebra took its own leading place in mathematics, people continued to derive insights and inspirations in beautiful constructions of elementary geometry.

The aim of this task is twofold. Firstly we would be happy to share with you the beauty of this eternal subject, and secondly we want to explain to you how simple concepts, which govern mathematics nowadays, work in every problem of elementary geometry. We wish you to enjoy this task and suggest that you use a pencil, a ruler, a compass, and a rubber to draw pictures for making your vision of the problems more precise.

2 Equivalence of Geometrical Figures

When we work on a geometrical problem we always use our own perception of the real world. After many years of successful (and sometimes not very successful) attempts people came to a conclusion that to work formally in geometry it is enough to follow a few standard rules to deduce any true statement about positions or configurations of geometrical bodies on a plane. Now these rules are called axioms and they strictly describe everything that you are allowed to do in geometry.

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Many years later when more sophisticated methods came to life in mathematics, it was realised that a global view on geometry gives a more productive way for solving problems than a step-by-step analysis based on abstract axioms.

At the heart of this appreciation there is the notion of an equivalence or an identity of geometrical figures. Assume that we have two geometrical figures drawn on a piece of paper, and we wish to decide if they are the same (equal) or not. We can cut both the figures out and try to cover one by the other. If it is possible to do this in such a way that every point of each of the figures is covered by exactly one point of the other, then we say that the figures are equal. We can think of this process of comparison as a move of one figure along the piece of paper and then a suitable rotation of it which makes the figures coincide. Let us now think how to make this explanation into a rigorous mathematical definition?

Before giving the formal definition we observe that there is something you are not allowed to do when you move figures. We cannot shrink or stretch anything, otherwise to say that figures are equal would make no sense at all. Recall that the distance between points on an Euclidean plane \mathbb{R}^2 is calculated in terms of coordinates in the Euclidean coordinate system (the system of coordinates with two orthogonal axes with the same unit of length along each of the axes). If a point A has coordinates (x_1, y_1) and a point B has coordinates (x_2, y_2) , then the distance between A and B is the length of the segment AB :

$$|AB| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Now we are ready to give our main definition.

Definition 2.1 A map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which preserves distance between any two points $A, B \in \mathbb{R}^2$:

$$|f(A)f(B)| = |AB|$$

is an isometry.

What are the typical examples of isometries? Well, there is a tautological one, namely, the identical transformation id (or just the identity) which leaves all points just where they were before:

$$id(A) = A, \quad \forall A \in \mathbb{R}^2.$$

An important property of the identity is that if you compose id with any other transformation R of the plane, you will get the same transformation R :

$$id \circ R = R \circ id = R.$$

Let us consider other types of isometries.

The parallel translation is an isometry which moves all points on the plane the same distance and in the same direction parallel to a fixed line.

The *rotation* is an isometry which keeps one point fixed (*the centre of the rotation*) and rotates all other points through a fixed *angle of the rotation* about the centre.

The *central symmetry* is an isometry which keeps one point O fixed and maps every point A to a point $B = \bar{B}(A)$ such that the fixed point O is the midpoint of the segment AB . Here the point O is the *centre of the symmetry*.

The last isometry we are going to mention is the *reflection in a line l (a mirror)* which maps every point A into a point $B = B(A)$ such that the line AB is orthogonal to the mirror and the intersection point of AB with the mirror is precisely the midpoint of the segment AB .

In principal, if we started from some fixed set of axioms we would have to prove that the transformations of the plane mentioned above are indeed isometries, i.e. that they preserve distances. Here we accept a different point of view on this matter. The fact that these transformations are isometries is experimentally verifiable, and it does reflect our perception of nature. Therefore, we postulate that *parallel translations, rotations, symmetries and reflections are isometries*.

The distance preserving property leads to various profound facts about isometries. For example:

Test Question 1 *Prove that if three points belong to a line, then under any isometry these points go into the points which again lie on a line.*

Definition 2.2 *Two figures Γ_1 and Γ_2 on a plane are equal if there is an isometry $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that the restriction of the map f on the set of points Γ_1 is a one-to-one correspondence $f : \Gamma_1 \rightarrow \Gamma_2$.*

Let us prove the following condition for triangles to be equal:

Theorem 2.1 *Two triangles ABC and $A'B'C'$ are equal if and only if three sides of the first triangle have the same lengths as the corresponding sides of the second triangle.*

Proof. This problem has two parts. First, we have to show that if there is an isometry which maps the triangle ABC into $A'B'C'$ then the sides of the triangles are equal. This is easy. We can see from Test Question 1 that under any isometry the lines which contain the sides of the triangle ABC are mapped into the lines which contain the sides of the triangle $A'B'C'$.

Test Question 2 *Prove that from here it follows that the vertexes A , B and C are mapped into the vertexes of the triangle $A'B'C'$.*

Because the distances between vertexes are preserved, the corresponding sides of the triangles are equal.

The second part of the problem is to construct an isometry which maps the triangle ABC into $A'B'C'$. We will use the following observation.

Test Question 3 *Prove that a composition of two isometries is again an isometry.*

We will try to reconstruct the desired isometry step-by-step. First we assume that the sides of the triangles are equal in the following order

$$|AB| = |A'B'|, \quad |BC| = |B'C'|, \quad |AC| = |A'C'|.$$

Step 1. We make a parallel translation along the line AA' and move all points on the distance $|AA'|$ so that the vertexes A and A' of the triangles coincide.

Step 2. We make a rotation about the point $A = A'$ through the angle α between the lines AB and $A'B'$ so that the points B and B' coincide.

Test Question 4 *Why is such a rotation possible? What is an axiom of Euclidean geometry we are using here?*

Step 3. If the points C and C' belong to the opposite half-planes with respect to the line $AB = A'B'$ we will apply the reflection in this line and move the point C into the same half-plane as the point C' .

Let us now check that the composition of *Step 1*, *Step 2* and *Step 3* maps the triangle ABC into $A'B'C'$ and the map is one-to-one. In fact, since each step is an isometry, no two points of ABC can be mapped into a single point, and we have already seen that the side AB is mapped into the side $A'B'$. Therefore, it is sufficient to check that C is mapped into C' . Let us draw two circles: one with the centre at $A = A'$, radius $|AC|$, and the other with the centre at $B = B'$, radius $|BC|$. Two distinct circles can have no more than two distinct common (intersection) points. Moreover, in each half-plane defined by the line AB there is at most one common point. The points C and C' belong to both of these circles and to the same half-plane. Therefore, $C = C'$ and the triangles coincide.

Test Question 5 *Using a coordinate system prove that two distinct circles can have no more than two intersection points.*

Test Question 6 *Formulate and prove two other conditions for equality of triangles: by two sides and an angle between them, and by two angles and a side.*

Recall that the medians of a triangle are the lines joining a vertex to the midpoint of the opposite side.

Example 2.1 *Corresponding medians of two triangles have equal lengths. Prove that the triangles are equal.*

Solution. Let us denote the first triangle by ABC and the second triangle by $A'B'C'$. Consider a central symmetry with respect to the midpoint of BC . It maps A into a point A_1 and AA_1 has length twice of the median m_a from A into BC . Similar, a symmetry with respect to the midpoint of BA_1 maps C into a point C_1 and $|CC_1| = 2|m_b|$, where m_b is the median of ABC from B into AC . A symmetry with respect to the midpoint of BC_1 maps point A_1 into a point A_2 such that $|A_1A_2| = 2|m_c|$, where m_c is the median from C into AB .

Test Question 7 Prove that the symmetry with respect to the midpoint of BA_2 maps C_1 into a point C_2 which is symmetric to C with respect to the midpoint of AB .

We can easily see that $|AA_2| = 2|m_b|$. The very same construction can be applied to $A'B'C'$ and we obtain a triangle $A'A'_1A'_2$ which has the same sides as AA_1A_2 . The isometry which maps AA_1A_2 into $A'A'_1A'_2$ maps ABC into $A'B'C'$. Therefore, ABC and $A'B'C'$ are equal.

Test Question 8 Explain why the isometry which maps AA_1A_2 into $A'A'_1A'_2$ also maps ABC into $A'B'C'$.

3 Translations

To define a parallel translation (or in short, a translation) we need the following data: a) a line l parallel to which we make translations, b) a direction on l in which we make translations and c) a distance d on which we translate every point. The simplest way to represent this data is in terms of vectors. A vector is a directed segment on a plane. When we speak about vectors we usually do not fix the end points of them. Any directed segment on a plane which is parallel to the same direction, has the same length and the same direction defines the same vector. We can draw a vector \vec{a} starting from any point A on a plane by choosing a line l through A parallel to the direction of the vector \vec{a} and drawing a point B on this line such that $|AB|$ equals to the length of the vector \vec{a} and the direction from A to B coincides with the direction of \vec{a} .

An important property of vectors is the possibility to take sums of vectors. It is in contrast to points and segments on plane, which you cannot add in any way. To define a vector $\vec{a} + \vec{b}$ we draw \vec{a} from any fixed point A , from the second endpoint B of \vec{a} we draw \vec{b} and obtain the second endpoint C of \vec{b} , then $\vec{a} + \vec{b} = \vec{AC}$. For the vector \vec{a} we define a parallel translation $T_{\vec{a}}$ as a transformation of a plane \mathbb{R}^2 that maps each point $A \in \mathbb{R}^2$ into the second endpoint of the vector \vec{a} drawn from the point A .

Test Question 9 Verify that $T_{\vec{a}} \circ T_{\vec{b}} = T_{\vec{b}} \circ T_{\vec{a}} = T_{\vec{a}+\vec{b}}$.

Let us have a look how parallel translations help to solve problems.

Example 3.1 A point M moves inside a triangle ABC starting from a point A_0 on AB parallel to the side BC until it meets the side AC . After that it moves parallel to AB until it meets the side BC , after that it moves parallel to AC until it meets with AB , etc. Prove that after a finite number of steps the trajectory of the point will repeat.

Solution. We denote the consecutive points of the trajectory of M which lie on the sides of the triangle by $A_0, A_1, B_1, B_2, C_2, C_3, A_3, A_4, B_4, \dots$. We know that the line A_1B_1 is parallel to $AB_2, B_1B_2 \parallel CA_1$ and $B_1C \parallel B_2C_2$. Therefore, the triangle AB_2C_2 is

obtain from A_1B_1C by a parallel translation, the triangle A_3BC_3 is obtained by a parallel translation from AB_2C_2 , and A_4B_4C by a parallel translation from A_3BC_3 . On the other hand it is easy to see that the triangle A_1B_1C is also a parallel translation of A_3BC_3 . The two images A_1B_1C and A_4B_4C of A_3BC_3 under these two parallel translations coincide because they both have the same vertex C (Why does this condition guarantees $A_1B_1C = A_4B_4C$?). Therefore, $A_4 = A_1$. We conclude that at most after seven steps the trajectory will come back to the starting point and will start to repeat itself.

Some geometrical problems require us to find a description for a set of points with a given property. To solve such problems means to find a recipe for constructing this set of points using ruler and compass. Here a knowledge of the properties of various transformations may play a crucial role as well.

Test Question 10 *Prove, that parallel translations maps circles into circles.*

Example 3.2 *Find the geometric locus of points with the sum of distances from two given intersecting lines equal to a given value t .*

Solution. Let X be a point which possesses the property required. Let us draw perpendiculars XA_1 and XA_2 from the point X to the given lines l_1, l_2 . Chose a point B on the ray A_1X (note on *the ray* not on *a line* or *a segment*) such that $|A_1B| = t$. Since $|XA_1| + |XA_2| = t$, then $|XB| = |XA_2|$. Let l'_1 be the image of l_1 by the parallel translation on vector $\overrightarrow{A_1B}$. Denote the intersection point of l'_1 and l_2 by M . Then under the assumption above, MX is the bisector of the angle $\angle A_2MB$.

To deduce the answer we need to take into account the fact that there are four different types of positions for the point X with respect to l_1 and l_2 . These two lines divide the plane into four regions. Depending on the region the place of the point B and of the ray A_1X will vary. Let us draw four lines l'_1, l''_1, l'_2, l''_2 such that l'_1 and l''_1 are parallel translations of l_1 by the distance t from l_1 ; and similar l'_2, l''_2 are parallel translations of l_2 by the distance t from l_2 . The set of points we are looking for are bisectors of angles between l_1 and l'_2, l_1 and l''_2, l_2 and l'_1, l_2 and l''_1 . The only tricky thing now is to chose the right angle out of four corresponding to each pair of lines.

Test Question 11 *Prove that the intersection points of l_1 with l'_2 and l''_2 , and of l_2 with l'_1 and l''_1 form a rectangle $M_1M_2M_3M_4$, whose sides are exactly the set of points with the required property.*

Test Question 12 *How can you reconstruct the rectangle $M_1M_2M_3M_4$ using ruler and compass, the two given lines l_1, l_2 and a segment of length t ?*

4 Central Symmetries

To define a central symmetry S_A we need only a single point A on a plane. Then S_A maps any point X into a point X' such that A is the midpoint of the segment XX' . This transformation may seem to be the simplest one but as we shall see in a moment, every parallel translation can be expressed as a composition of two central symmetries. To start with, let us observe that $S_A \circ S_A = id$ is the identical transformation (all points return to their original position after double application of a central symmetry, i.e. $S_A \circ S_A$ does not change anything).

Proposition 4.1 *A composition of two central symmetries and a composition of a central symmetry with a parallel translation can be calculated via formulas*

$$S_B \circ S_A = T_{\vec{a}}; \quad T_{\vec{a}} \circ S_A = S_B \quad S_B \circ T_{\vec{a}} = S_A,$$

where $\vec{a} = 2\overrightarrow{AB}$.

Proof. First we prove that $S_B \circ S_A = T_{\vec{a}}$. Let a point C go into a point C_1 under the central symmetry S_A and C_1 go into C_2 under the central symmetry S_B . Then AB is the mid-line of the triangle CC_1C_2 and, therefore, $\overrightarrow{CC_2} = 2\overrightarrow{AB} = \vec{a}$.

To prove the second and the third formulas we observe that the point B is the image of A under a parallel translation on $\vec{a}/2$. As we have just proved $T_{\vec{a}} = S_B \circ S_A$. Therefore,

$$T_{\vec{a}} \circ S_A = (S_B \circ S_A) \circ S_A = S_B \circ (S_A \circ S_A) = S_B \circ id = S_B.$$

The third formula follows similarly.

Example 4.1 *A point A is mapped into A_1 by a central symmetry with respect to a point O_1 , then A_1 is mapped into A_2 by a central symmetry with respect to a point O_2 , then A_2 is mapped into A_3 by a central symmetry with respect to O_3 . Then everything is repeated starting with the point A_3 :*

$$A_4 = S_{O_1}(A_3), \quad A_5 = S_{O_2}(A_4), \quad A_6 = S_{O_3}(A_5).$$

Prove that $A_6 = A$.

Solution. From Proposition 4.1 we know that for any two points A and B

$$S_B \circ S_A = T_{2\overrightarrow{AB}}$$

From here we deduce that

$$S_{O_3} \circ S_{O_2} \circ S_{O_1} \circ S_{O_3} \circ S_{O_2} \circ S_{O_1} = (S_{O_3} \circ S_{O_2}) \circ (S_{O_1} \circ S_{O_3}) \circ (S_{O_2} \circ S_{O_1}) =$$

$$= T_{\overrightarrow{O_2O_3}} \circ T_{\overrightarrow{O_3O_1}} \circ T_{\overrightarrow{O_1O_2}} = T_{\overrightarrow{O_2O_3 + O_3O_1 + O_1O_2}}$$

which is the identical transformation as $\overrightarrow{O_2O_3} + \overrightarrow{O_3O_1} + \overrightarrow{O_1O_2} = 0$. Therefore,

$$A_6 = S_{O_3} \circ S_{O_2} \circ S_{O_1} \circ S_{O_3} \circ S_{O_2} \circ S_{O_1}(A) = id(A) = A.$$

Arguments by symmetry very often help to solve non-geometrical problems. Have a look:

Example 4.2 *Two players in turn put two-pence coins on a rectangle table. They can put coins only on free places and are not allowed to cover other coins. The loser is the one who cannot make the next step. Prove that the first player always has a winning strategy.*

Solution. The following strategy allows the first player to win. On the first step the first player puts a coin in the centre of the table. Then at each step the first player puts a coin on the place which is centrally symmetric to the place where the second player has just put a coin.

Test Question 13 *Why is this place always free under this strategy?*

Example 4.3 *Two circles S_1 and S_2 of radius 1 are touching at a point A . The centre O of a circle S of radius 2 belongs to S_1 . The circle S_1 touches S at a point B . Prove that the line AB pass through a common point of S_2 and S .*

Solution. The circles S_1 and S_2 are symmetric with respect to the point A . Because OB is a diameter, the angle $\angle BAO = 90^\circ$. Therefore, under the central symmetry with respect to A the image of the point B stays on the circle S (Why?). This implies that under this symmetry the point B goes into a common point C of S_2 and S , i.e. the points A , B and C belong to the same line.

5 Reflections in a line

The reflection S_l in a line l is an isometry which takes every point into its mirror image with l as a mirror. The image of a point X by the reflection in a line l is a point X' such that XX' is orthogonal to l and l intersects XX' in the midpoint of the segment XX' . Let us discuss properties of this transformation. Obviously, $S_l \circ S_l = id$.

Test Question 14 *Show that if a line l_1 is parallel to a line l_2 then $S_{l_1} \circ S_{l_2} = T_{2\vec{a}}$, where $T_{\vec{a}}$ is a parallel translation which maps l_2 into l_1 .*

Example 5.1 *Let $l_3 = S_{l_1}(l_2)$, then $S_{l_3} = S_{l_1} \circ S_{l_2} \circ S_{l_1}$.*

Solution. Consider a point X and its mirror image Y in the line l_3 :

$$S_{l_3}(X) = Y.$$

Because

$$S_{l_1}(l_3) = S_{l_1}(S_{l_1}(l_2)) = (S_{l_1} \circ S_{l_1})(l_2) = l_2,$$

the point $S_{l_1}(Y)$ is the mirror image of $S_{l_1}(X)$ in the line $l_2 = S_{l_1}(l_3)$, i.e.

$$S_{l_1}(X) = S_{l_2} \circ S_{l_1}(Y).$$

Therefore,

$$S_{l_1} \circ S_{l_3} = S_{l_2} \circ S_{l_1},$$

and

$$S_{l_3} = (S_{l_1} \circ S_{l_1}) \circ S_{l_3} = S_{l_1} \circ (S_{l_1} \circ S_{l_3}) = S_{l_1} \circ S_{l_2} \circ S_{l_1}.$$

One of the uses of the reflection is for solutions of geometric inequalities.

Example 5.2 Prove that the area of a convex quadrangle $ABCD$ is less than or equal to

$$\frac{|AB| \cdot |CD| + |AD| \cdot |BC|}{2}.$$

Solution. Let l be the line perpendicular to AC and passing through its midpoint, and let D' be the mirror image of D in l . Then

$$\begin{aligned} S_{ABCD} &= S_{ABCD'} = S_{BAD'} + S_{BCD'} = \\ &= \frac{|AB| \cdot |AD'| \cdot \sin \angle BAD'}{2} + \frac{|BC| \cdot |CD'| \cdot \sin \angle BCD'}{2} \leq \\ &\leq \frac{|AB| \cdot |AD'|}{2} + \frac{|BC| \cdot |CD'|}{2} = \frac{|AB| \cdot |CD| + |AD| \cdot |BC|}{2}. \end{aligned}$$

Example 5.3 Using a compass and a ruler reconstruct a triangle ABC if you are given a segment of length $b = |AC|$, a segment of length $a - b$ ($a > b$, $a = |CB|$), and an angle equal to $\angle C = \angle BCA$.

Solution. Let us assume that the triangle ABC is constructed. Let C' be a mirror image of A in the bisector of $\angle C$. Then the point C' is on the side CB and

$$\angle BC'A = 180^\circ - \angle AC'C = 180^\circ - \left(90^\circ - \frac{\angle C}{2}\right) = 90^\circ + \frac{\angle C}{2}.$$

Moreover, $|BC'| = a - b$. Therefore, using given values c , $a - b$ and $\angle C' = 90^\circ + \angle C/2$ we can reconstruct the triangle ABC' (How can you build an angle of $90^\circ + C/2$ using ruler and compass from the angle $\angle C'$?). To get the point C we have to draw the midpoint perpendicular to the segment AC' and continue it till the intersection point with the line containing BC' .

6 Orientation

Before we move further and study the remaining isometry — rotation, we have to discuss a very important notion which turns up in every serious geometric problem. This is the notion of orientation. When we speak about a line, a plane or a three-dimensional space, we always rely in our arguments (implicitly or explicitly) on understanding what is left and what is right, what is clockwise and what is anticlockwise etc. This is because we often need to distinguish figures which are mirror images of each other and we do not have any reason to prefer one of them to the other.

Definition 6.1 *An orientation of a line at a point P is a fixed way to define what is the left and what is the right direction on the line in the vicinity of P .*

Obviously, there are exactly two ways to define orientation at each point on a line. The following definition may look ridiculous at first glance but it gives the only way of generalisation for the notion of orientation in higher dimensions.

Definition 6.2 *An orientation of a line is a continuous choice of orientations at each point on the line.*

Proposition 6.1 *There are two possible orientations of a line.*

Solution. We can think of an orientation at a point as an arrow showing the direction. A continuous choice of orientations will give us a picture where all sufficiently close arrows point in the same direction. Because from every point on the line we can get to any other point on the line in, probably, a large number of sufficiently small steps, all arrows point in the same direction. There are only two directions for arrows, therefore, there are at most two orientations of a line. Obviously, either of them is possible.

To define an orientation of a plane we have to use the notions of clockwise and anticlockwise instead of right and left. People invented clocks many years ago and the prototype for the most widespread form of clocks was a sundial (many Cambridge Colleges use them up this day). This device is simple and accurate. When the sun is shining, a pointed piece of metal throws a shadow on a flat surface that is marked with the hours like clock and the shadow moves around as the sun moves across the sky. Later when mechanical clocks were invented, clock masters let all clock hands follow the usual direction of the shade. This direction is known as the clockwise direction.

But life could be a lot different if a sundial had been invented in the opposite (southern) hemisphere.

Test Question 15 *Explain why the shade of a stick at the South Pole moves anticlockwise.*

Definition 6.3 *An orientation of a plane at a point P is a fixed way to define what is the clockwise and what is the anticlockwise direction on the plane in the vicinity of P .*

Definition 6.4 *An orientation of a plane is a continuous choice of orientations at each point of the plane.*

Test Question 16 *Prove that there are only two possible orientations on a plane.*

It would be easy to suggest that the standard way of choosing the orientation of a piece of paper representing a plane is to set the clockwise direction looking someone's watch or at a nearby clock. In fact the situation is completely opposite, and a particular importance of this is the commonly accepted way of measuring angles. Two intersecting lines l_1 and l_2 form four angles (opposite angles have the same measure). The angle between l_1 and l_2 is one of these four angles chosen in such a way that the rotation from l_1 to l_2 inside this angle is an anticlockwise rotation (it does not actually matter which one of the two equal angles to chose from, the definition is correct anyway). In particular, this indicates that the usual orientation of a plane is opposite to the one given by clock hands.

We finish this section with a very important observation. Every transformation of a plane can change or keep unchanged the original orientation. By the change of an orientation we mean that angles between two lines l_1 and l_2 can be changed by a transformation R if we consider all four lines $l_1, l_2, R(l_1)$ and $R(l_2)$ to be lying on the same plane with a fixed orientation. We say that an isometry R preserves the orientation if for any lines l_1 and l_2

$$\angle(l_1, l_2) = \angle(R(l_1), R(l_2)),$$

otherwise we say that R changes orientation. In the later case

$$\angle(l_1, l_2) + \angle(R(l_1), R(l_2)) = 180^\circ$$

for every two intersecting lines l_1 and l_2 . Examples of transformations preserving the orientation are the parallel translation, the central symmetry and the rotation. The reflection in a line does not preserve the orientation.

Example 6.1 *Prove that the reflection in a line cannot be represented as a composition of various parallel translations and central symmetries.*

Solution. We know that the parallel translation and the central symmetry preserve the orientation. Therefore, so does any composition of them. But the reflection in a line does not preserve the orientation. Thus, the required composition is impossible.

7 Rotations

A rotation R_O^α is defined by an angle α of the rotation and a centre O of the rotation. We assume that all rotations considered are anticlockwise (this is consistent with our way of measuring angles between lines). The image of a point Y under a rotation R_O^α is the uniquely defined point X such that $|OX| = |OY|$ and the anticlockwise angle between the rays OY and OX is equal to α . We allow negative values for α which correspond to rotations in the clockwise direction. We also consider rotations through angles greater than 360° . The first property of rotations we are going to discuss is their relation to the reflection in a line.

Test Question 17 *Prove that if lines l_1 and l_2 intersect at a point O , then $S_{l_2} \circ S_{l_1} = R_O^{2\alpha}$, where R_O^α is the rotation which maps l_1 into l_2 .*

Observe now that

$$R_O^\alpha \circ R_O^\beta = R_O^{\alpha+\beta}, \quad R_O^{\alpha+360^\circ} = R_O^\alpha.$$

In fact a more general statement about rotations is valid.

Example 7.1 *A composition of two rotations the sum of whose angles is not a multiple of 360° , is again a rotation. Where is the centre of this rotation? What is the angle of this rotation?*

Solution. Let us denote the first rotation by R_A^α and the second one by R_B^β . If $A = B$, then the statement was discussed above. Therefore, we assume that $A \neq B$. Let l be the line which joins A and B . Consider a line a through A and a line b through B such that

$$\angle(a, l) = \alpha/2, \quad \angle(l, b) = \beta/2.$$

Using Test Question 17

$$R_B^\beta \circ R_A^\alpha = (S_b \circ S_l) \circ (S_l \circ S_a) = S_b \circ S_a.$$

If $a \parallel b$ then $S_a \circ S_b = T_{2\vec{u}}$, where $T_{\vec{u}}$ is the parallel translation which maps b to a in the direction perpendicular to a . If a and b are not parallel and $O = a \cap b$ is the intersection point, then $S_a \circ S_b$ is the rotation through $\alpha + \beta$ around O . It can easily be seen that $a \parallel b$ if and only if for some integer k

$$\frac{\alpha}{2} + \frac{\beta}{2} = k\pi, \quad \text{i.e.} \quad \alpha + \beta = 2k\pi.$$

Answer: The angle of the rotation is $\alpha + \beta$, the centre is the intersection point of the lines a and b constructed above.

Now we shall use the technique developed to prove a wonderful mathematical fact which is called Napoleon theorem. It is hardly possible that this famous person was such a genius in maths (as he was on the battlefield) as to prove such a theorem. Nevertheless his known passion for mathematics means this nice result is named after him.

Theorem 7.1 *If three equilateral triangles A_1BC , B_1AC and C_1AB are erected externally on the sides of a triangle ABC (with the corresponding length of sides $|BC|$, $|AC|$ and $|AB|$), then their centres O_A , O_B and O_C form an equilateral triangle.*

Before starting the proof of this remarkable theorem we discuss a simpler fact about the very same picture described in Theorem 7.1.

Example 7.2 *In the notations of Theorem 7.1 prove that $|AA_1| = |BB_1| = |CC_1|$ and that the lines AA_1 , BB_1 , CC_1 pass through the same point.*

Solution. Assume that the vertexes A , B , C of the triangle are positioned clockwise. The rotation $R_C^{\pi/3}$ moves the point A to B_1 and point A_1 to B . Therefore, $|AA_1| = |BB_1|$ and the angle between the lines AA_1 and BB_1 is 60° . Let P be the intersection point of AA_1 and BB_1 . Because $\angle BCA_1 = \angle(AA_1, BB_1) = 60^\circ$, the point P belongs to the circle circumscribed around BCA_1 . Similar, P belongs to the circles circumscribed around ABC_1 and ACB_1 . This defines the point P as the unique intersection point of three circumcircles of the equilateral triangles ABC_1 , ACB_1 and BCA_1 .

Now we can repeat this argument changing A_1 , B_1 to B_1 , C_1 . We obtain that $|BB_1| = |CC_1|$ and the intersection point of BB_1 and CC_1 belongs to each of the three circumcircles of the equilateral triangles ABC_1 , ACB_1 and BCA_1 , i.e. it coincides with P . Thus all three lines AA_1 , BB_1 , CC_1 meet at P .

Proof of Theorem 7.1. We shall make use of Example 7.1 which says that a composition of two rotations through angles α and β is again a rotation through angle $\alpha + \beta$ unless $\alpha + \beta = 2\pi k$, $k \in \mathbb{Z}$, in which case the composition is a parallel translation.

Consider a composition Q of rotations $R_{O_A}^{2\pi/3}$, $R_{O_B}^{2\pi/3}$, $R_{O_C}^{2\pi/3}$. We apply Q to the point B :

$$Q(B) = R_{O_C}^{2\pi/3} \circ R_{O_B}^{2\pi/3} \circ R_{O_A}^{2\pi/3}(B) = R_{O_C}^{2\pi/3} \circ R_{O_B}^{2\pi/3}(C) = R_{O_C}^{2\pi/3}(A) = B,$$

i.e. B is a fixed point for Q : $Q(B) = B$. We know that $R_{O_B}^{2\pi/3} \circ R_{O_A}^{2\pi/3}$ is a rotation through the angle 240° around an unknown point X . As $120^\circ + 240^\circ = 360^\circ$, from Example 7.1 it follows that

$$Q = R_{O_C}^{2\pi/3} \circ \left(R_{O_B}^{2\pi/3} \circ R_{O_A}^{2\pi/3} \right)$$

is a parallel translation. But the only parallel translation which has a fixed point is the identical transformation. Therefore, $Q \equiv id$ and

$$R_{O_C}^{-2\pi/3} = R_{O_C}^{-2\pi/3} \circ Q = \left(R_{O_C}^{-2\pi/3} \circ R_{O_C}^{2\pi/3} \right) \circ \left(R_{O_B}^{2\pi/3} \circ R_{O_A}^{2\pi/3} \right) = \left(R_{O_B}^{2\pi/3} \circ R_{O_A}^{2\pi/3} \right).$$

From here

$$Y = R_{O_C}^{-2\pi/3}(O_A) = R_{O_B}^{2\pi/3} \circ R_{O_A}^{2\pi/3}(O_A) = R_{O_B}^{2\pi/3}(O_A).$$

If we look at the quadrangle $O_C O_A O_B Y$ we find that

$$\angle O_A O_C Y = \angle O_A O_B Y = \frac{2\pi}{3}, \quad |O_C O_A| = |O_C Y|, \quad |O_B O_A| = |O_B Y|.$$

This implies that $\angle O_C O_A O_B = \angle O_C Y O_B = 60^\circ$. In the same vein we can check that $\angle O_A O_B O_C = \angle O_A O_C O_B = 60^\circ$, i.e. the triangle $O_A O_B O_C$ is equilateral.

We shall finish this section with an example in which you cannot trivially recognise an application of rotations.

Example 7.3 *A lion is running along the circus arena which is a circle with radius 10 meters. The route of the lion is a broken line of the total length 30km. Prove that the sum of (non-oriented, i.e. $\leq \pi$) angles of all lions turns is at least 2998 (calculated in radian measure).*

Solution. We assume that the broken line which forms the trajectory of the line has vertexes A_1, A_2, \dots, A_n . Let us “straighten” the trajectory in the following way. We rotate each point $A_j, j = 3, 4, \dots, n$, together with the remaining part of the trajectory $A_{j+1} \dots A_n$ through an angle of the turn the lion made at point A_j in the opposite direction, i.e. A_3 will be moved onto the ray $A_1 A_2$, then A_4 will be moved onto the ray $A_1 A_2$, etc., so that the images of A_1, \dots, A_n lie in a straight line. Consider the consecutive images of the centre O of the arena under the rotations described above. First O will be moved to a point O_2 , then from the point O_2 it goes into a point O_3 , etc, up to the final point O_{n-1} (we will also think of O as O_1). The images of A_1, A_2, \dots, A_n under these rotations we denote by A'_1, A'_2, \dots, A'_n , and these points lie in the same line which is the ray $A_1 A_2$.

Let α_{i-1} be the angle of lion’s turn at point A_i . Then $\angle O_{i-1} A'_i O_i = \alpha_{i-1}$ and $|A'_i O_{i-1}| = |A'_i O_i| \leq 10$. From here it follows that $|O_i O_{i-1}| \leq 10\alpha_{i-1}$.

Test Question 18 *Explain why for a triangle ABC with $|AB| = |BC|$ and $\angle ABC = \beta$, $\beta \in [0, \pi]$ it is true that*

$$|AC| \leq |AB| \cdot \beta.$$

Therefore,

$$\begin{aligned} 30000 &= |A_1 A_2| + \dots + |A_{n-1} A_n| = |A'_1 A'_2| + \dots + |A'_{n-1} A'_n| = \\ &= |A'_1 A'_n| \leq |A'_1 O_1| + |O_1 O_2| + \dots + |O_{n-2} O_{n-1}| + |O_{n-1} A'_n| \leq \\ &\leq 10 + 10(\alpha_1 + \dots + \alpha_{n-2}) + 10, \end{aligned}$$

i.e. $(\alpha_1 + \dots + \alpha_{n-2}) \geq 2998$.

8 Classification of isometries on a plane

The last topic we discuss in this task is the complete classification of isometries on a plane. Looking back to the types of isometries studied in the previous sections, one may hope that other examples exist. But let us try to understand how much information about an isometry we need to determine it uniquely.

Definition 8.1 Two isometries P_1 and P_2 of a plane are called equal ($P_1 \equiv P_2$) if for every point A on a plane $P_1(A) = P_2(A)$.

Observe that every isometry $P : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has an inverse $P^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. The inverse is an isometry which maps a point $B = P(A)$ back into A . Obviously, the inverse has the following property:

$$P^{-1} \circ P = P \circ P^{-1} = id.$$

Test Question 19 Prove that if there is a transformation $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that for an isometry P at least one of the identities holds:

$$A \circ P = id \quad P \circ A = id,$$

then A is an isometry and $A = P^{-1}$.

Test Question 20 Show that an isometry P_1 is equal to P_2 if and only if $P_1^{-1} \circ P_2 \equiv id$, i.e. that

$$P_1^{-1} \circ P_2(A) = A, \quad \forall A \in \mathbb{R}^2.$$

Theorem 8.1 Let P_1 and P_2 be two isometries of a plane. If for three points A , B and C which are in general position on a plane (do not lie on the same line), we have that

$$P_1(A) = P_2(A), \quad P_1(B) = P_2(B), \quad P_1(C) = P_2(C),$$

then P_1 is equal to P_2 , i.e. for any point D on a plane $P_1(D) = P_2(D)$.

Proof. As we discussed earlier we have to show that $P_1^{-1} \circ P_2(D) = D$. Due to

$$P_1(A) = P_2(A), \quad P_1(B) = P_2(B), \quad P_1(C) = P_2(C),$$

we know that

$$P_1^{-1} \circ P_2(A) = A \quad P_1^{-1} \circ P_2(B) = B, \quad P_1^{-1} \circ P_2(C) = C.$$

Because $P_1^{-1} \circ P_2$ is an isometry the distances from the point $P_1^{-1} \circ P_2(D)$ to the points $P_1^{-1} \circ P_2(A)$, $P_1^{-1} \circ P_2(B)$, $P_1^{-1} \circ P_2(C)$ are the same as from the point D to A , B and C respectively. Therefore, both D and $P_1^{-1} \circ P_2(D)$ belong to the intersection of three

circles of radius $|AD|$, $|BD|$ and $|CD|$ with centres at A , B and C respectively. Because three circles whose centres are not on the same line, can have no more than one common point, we deduce that

$$P_1^{-1} \circ P_2(D) = D.$$

From this theorem we see that to reconstruct an isometry uniquely we only need to know three images of vertexes of any triangle. This is the time to look back at the way we proved one of the conditions for equality of triangles.

Theorem 8.2 *Any isometry on a plane is a composition of a translation, a rotation and possibly of a reflection in a line.*

Proof. Consider an isometry P and a triangle ABC on a plane. We know that the triangles ABC and $A'B'C'$, where $A' = P(A)$, $B' = P(B)$ and $C' = P(C)$, are equal. Let us look at the composition of a translation $T_{\vec{a}}$ on the vector $\vec{a} = \overrightarrow{AA'}$ and a rotation $R_{A'}^\alpha$ through the angle $\alpha = \angle B'A'K$, $K = T_{\vec{a}}(B)$. Two situations are possible:

a) the points C' and $R_{A'}^\alpha \circ T_{\vec{a}}(C)$ belong to the same half-plane with respect to the line $A'B'$;

b) the points C' and $R_{A'}^\alpha \circ T_{\vec{a}}(C)$ belong to the different half-planes with respect to the line $A'B'$.

In the case a) we know from the proof of Theorem 2.1, that

$$C' = R_{A'}^\alpha \circ T_{\vec{a}}(C)$$

and in the case b) we know that

$$C' = S_{A'B'} \circ R_{A'}^\alpha \circ T_{\vec{a}}(C),$$

where as usual $S_{A'B'}$ is the reflection in the line $A'B'$. Therefore, every isometry is either a composition of a translation and a rotation or a composition of a translation, a rotation and a reflection in a line.

There is another way to decompose every isometry into a composition of elementary isometries. This is the so-called Chasles theorem:

Theorem 8.3 *Every isometry is a composition of at most three reflections in lines.*

Test Question 21 *Prove this theorem. Hint: you have to construct a sequence of at most three reflections in lines which maps a given triangle ABC into an equal one $A'B'C'$. As we did earlier, try to construct this sequence in consecutive steps where the first maps A to A' , the second maps B to B' , and, finally, the third maps C to C' .*

Conclusion

We suggest you to try to solve the problem sheet below which contains problems whose solution may be simplified significantly if you notice an isometry that helps to describe the picture (do use a compass and a ruler to draw everything). Nevertheless, we do not insist on your using only methods related to those discussed in the task. Every path to the solution is good and, for example, in Problem 30 we do not know a solution based on isometries (a special prize for those who find one). The only requirement we ask you to follow is to explain solutions at the same level of rigorousness as was done throughout the task. Good luck!

9 Problem Sheet.

Problem 1 *Answer the test questions in the text.*

Problem 2 *Prove the following condition for equality of tetrahedrons in the three-dimensional Euclidean space: two tetrahedrons are equal if and only if lengths of their corresponding edges are equal.*

Problem 3 *What is the place to build a bridge MN across a river which separates two villages A and B , so that the route $AMNB$ will be of the shortest possible length? We assume that the river banks are parallel and the bridge is perpendicular to the river banks.*

Problem 4 *In a trapezium $ABCD$ the sides BC and AD are parallel. Let M be the intersection point of the bisectors of the angles A and B , and N be the intersection point of the bisectors of the angles C and D of the trapezium. Prove that*

$$2|MN| = |AB + CD - BC - AD|.$$

Problem 5 *Two altitudes BK and BH of a parallelogram $ABCD$ are drawn from the vertex B on the sides AD and CD respectively. Lengths $|KH| = a$ and $|BD| = b$ are known. Find the distance from the point B to the orthocenter of the triangle BKH .*

Problem 6 *A circle meets sides BC , CA and AB of a triangle ABC at points A_1 and A_2 , B_1 and B_2 , C_1 and C_2 respectively. Prove that if perpendiculars drawn to the sides of the triangle at points A_1 , B_1 and C_1 , meet at the same point, then perpendiculars to the sides of the triangle drawn at points A_2 , B_2 and C_2 also meet at the same point.*

Problem 7 *Prove that perpendiculars to the sides of a quadrilateral inscribed in a circle, drawn through the midpoints of the opposite sides, meet at the same point.*

Problem 8 *Let P be the midpoint of the side AB of a convex quadrilateral $ABCD$. Prove that if the area of the triangle PCD is equal to half of the area of $ABCD$ then $BC \parallel AD$.*

Problem 9 Consider n pairs of points on a segment AB , such that points in each pair are symmetric with respect to the midpoint of AB . One half of these $2n$ points are coloured red and the other half are coloured blue. Prove that the sum of distances from A to the red points is equal to the sum of distances from B to the blue points.

Problem 10 Using ruler and compass, draw a segment with endpoints on the sides of a given angle such that a given point inside the angle is the midpoint of the segment. Is a segment with such properties unique?

Problem 11 A point M lies on a diameter AB of a circle S . A chord CD goes through M and intersects AB under the angle 45° . Prove, that

$$f(M) = |CM|^2 + |DM|^2$$

is a constant function, i.e. its value does not depend on the choice of $M \in AB$.

Problem 12 The incircle of the triangle ABC meets the sides AC and BC at points B_1 and A_1 respectively. Prove that if $AC > BC$ then $AA_1 > BB_1$.

Problem 13 By means of a ruler and a compass reconstruct a triangle using two of its sides $a = |BC|$, $b = |AC|$ and the difference between angles $\angle A$ and $\angle B$ ($\angle A > \angle B$).

Problem 14 Let a, b, c be three lines on a plane. Denote the composition $S_a \circ S_b \circ S_c$ by T . Prove that $T \circ T$ is a parallel translation (or the identical transformation).

Problem 15 The incircle of the triangle ABC is tangent to the sides of the triangle at points A_1, B_1 and C_1 ($A_1 \in BC, B_1 \in AC, C_1 \in AB$). Let A_2, B_2, C_2 be mirror images of A_1, B_1, C_1 in the bisectors of the corresponding angles $\angle A, \angle B, \angle C$ of ABC . Prove that $A_2B_2 \parallel AB$ and the lines AA_2, BB_2, CC_2 meet at the same point.

Problem 16 A figure on a plane is preserved by reflections in two distinct lines and does not have any other mirrors. Prove that the mirror-lines are perpendicular.

Problem 17 Two points M and K are chosen on the sides BC and CD respectively of a square $ABCD$, in such a way that $\angle BAM = \angle MAK$. Prove that $|BM| + |KD| = |AK|$.

Problem 18 A point P is chosen inside a square $A_1A_2A_3A_4$. We draw perpendiculars from A_1 on A_2P , from A_2 on A_3P from A_3 on A_4P , and from A_4 on A_1P . Prove that all the four perpendiculars meet at the same point.

Problem 19 Points M and K are chosen on the sides CB and CD of a square $ABCD$ in such a way that the perimeter of CMK is twice the length of a side of the square. Find the angle $\angle MAK$.

Problem 20 Two squares $ABMN$ and $BCPQ$ are erected externally on the sides AB and BC of a triangle ABC . Prove that the centres of these squares and the midpoints of MQ and AC are four vertices of a square.

Problem 21 Two equilateral triangles ABC and CDE are erected in the same half-plane on a segment AE , so that the point C lies on AE . Let M and P be the midpoints of AD and BE respectively. Prove that the triangle CPM is equilateral.

Problem 22 Let $ABCD$ be a parallelogram, BCP and CDQ are equilateral triangles erected externally on $ABCD$. Prove that the triangle APQ is also equilateral.

Problem 23 A point M lies on an arc AB of a circumcircle of an equilateral triangle ABC . Prove that $MC = MA + MB$.

Problem 24 A rotation about a point O maps a line l_1 into l_2 and a point $A_1 \in l_1$ into $A_2 \in l_2$. Prove that the intersection point of l_1 and l_2 belongs to the circumcircle of the triangle A_1OA_2 .

Problem 25 Four squares are erected externally on the the sides of a convex quadrangle. Prove that the lines connecting centres of opposite squares are perpendicular.

Problem 26 Three squares with the centres at points P , Q and R are erected externally on the sides of a triangle ABC . Then another three squares are erected internally on the sides of PQR . Prove that the centres of the last three squares are the midpoints of the sides of ABC .

Problem 27 Prove the second part of the Napoleon theorem which says that the centres of the equilateral triangles erected internally on the sides of any triangle form an equilateral triangle (the inner Napoleon triangle).

Problem 28 Prove that the difference between areas of the outer and inner Napoleon triangles equals the area of the original triangle.

Problem 29 Prove that every isometry which preserves orientation is a composition of two reflections in lines.

Problem 30 Consider trisectors of the three angles of a triangle ABC . Two trisectors nearest to the side BC meet at a point A_1 , two trisectors nearest to the side AC meet at a point B_1 , and the remaining two trisectors meet at a point C_1 . Prove that the triangle $A_1B_1C_1$ is equilateral.

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- [2] V. V. Prasolov, *Problems on Plane Geometry*, MCCME, 2001.

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