

**EXISTENCE OF PARTITIONS OF UNITY  
PART III DIFFERENTIAL GEOMETRY 2011**

(The following is a technical tool, that for this course is non-examinable in the non-compact case)

**Theorem 1.** *Let  $M$  be a manifold. Then given any open cover  $\{V_\beta\}$  there exists a partition of unity  $\{\phi_i\}$  subordinate to  $\{V_\beta\}$ .*

*Proof.* Recall that a topological space is said to be *locally compact* if every point is contained in an open set whose closure is compact. Clearly Euclidean space is locally compact, and as  $M$  is locally homeomorphic to Euclidean space and it is also locally compact.

Now we use the assumption that  $M$  is second countable, so there is a countable basis of open sets  $\{B_j : j \in \mathbb{N}\}$ . By local compactness, any  $x \in M$  is contained in some open  $U_x$  whose closure is compact. Thus there is a  $j(x)$  such that  $x \in B_{j(x)} \subset U_x$ , and in particular  $\overline{B_{j(x)}} \subset \overline{U_x}$ . Now a closed subset of a compact set is compact, so  $B_{j(x)}$  has compact closure. Hence by shrinking our countable basis if necessary, we may assume that each  $B_j$  has compact closure.

Now define  $W_1 = B_1$ . Then by compactness

$$\overline{W_1} \subset \bigcup_{j=1}^m B_j$$

for some  $m$ . Next define  $W_2$  to be the union  $W_2 = \bigcup_{j=1}^m B_j \cup B_2$ . Then  $\overline{W_2}$  is a union of open sets which have compact closure, and thus is also open with compact closure. Repeating the above we get an open cover  $W_j$  of sets which each have compact closure and such that  $\overline{W_j} \subset W_{j+1}$ . In particular then

$$\overline{W_j}/W_{j-1} \subset W_{j+1}/\overline{W_{j-2}}. \tag{2}$$

[see the remark below for an explanation of what is going on]

Now fix  $p \in M$  and let  $j$  be the largest natural number so  $p \in M/\overline{W_j}$ . Then  $p \in V_\beta \cap (W_{j+2}/\overline{W_{j-1}})$  for some  $\beta$ . Take a chart  $U_p$  contained this open set and let  $f$  be a bump function which is identically 1 on an neighbourhood  $N_p$  of  $p$  and whose support is within this chart. Now as  $p$  ranges over  $W_{j+2}/\overline{W_{j-1}}$ , the  $N_p$  cover  $\overline{W_{j+1}}/W_j$  so by compactness we can take a finite subcover. Thus there exist a finite number of bump functions  $f_k$  with the property that at least one of them does not vanish at any given point in  $\overline{W_{j+1}}/W_j$  and such that each  $f_k$  has support contained in  $V_{\beta(k)} \cap (W_{j+2}/\overline{W_{j-1}})$  for some  $\beta(k)$ . Repeating this for all  $j$  we are left with a countable collection of bump function  $\{\phi_i\}$  whose supports are locally finite and such that at every point at least one does not vanish. Thus  $\psi_i := \phi_i / \sum_u \phi_u$  is a well-defined partition of unity subordinate to  $\{V_\beta\}$ . □

**Remark.** Some of you may have heard of Urysohn's metrization theorem which says that any regular second countable topological space is metrizable. Applying this to our manifold  $M$  we obtain a metric  $d$  that defines the topology (I know of no other place where this metric is in the last bit useful for the study of manifolds; in fact for metric properties one usually assumes there is a Riemannian metric which will appear later in the course). However given  $d$  one can define the open sets  $W_j$  more easily by fixing a base point  $p \in M$  and letting  $W_j := \{x \in M : d(x, p) < j\}$ . Clearly this is open and has compact closure, and (2) is obvious.

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