THE LEVI-CIVITA CONNECTION
PART III DIFFERENTIAL GEOMETRY 2011

Theorem 1. *Let M be a manifold and g be a Riemannian metric on M. Then there exists a unique connection ∇ on M that is symmetric and compatible with g.*

Proof. We first show uniqueness. Suppose such a ∇ existed and let X, Y, Z be vector fields on M. We write $\langle X, Y\rangle$ for the inner product $g(X, Y)$. Symmetry implies $\nabla_X Z = \nabla_Z X + [X, Z]$. Using this and compatibility of ∇ with g we have

$$\nabla_X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X, Z]\rangle$$

By swapping X, Y, Z around we thus have

$$\begin{aligned}\nabla_X\langle Y, Z\rangle &= \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X, Z]\rangle\\ \nabla_Y\langle Z, X\rangle &= \langle\nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle + \langle Z, [Y, X]\rangle\\ \nabla_Z\langle X, Y\rangle &= \langle\nabla_Z X, Y\rangle + \langle X, \nabla_Y Z\rangle + \langle X, [Z, Y]\rangle\end{aligned}$$

Adding the first two and subtracting the third gives and equation for $\langle\nabla_X Y, Z\rangle$, which after some rearranging becomes

$$\langle\nabla_X Y, Z\rangle = \frac{1}{2}\left(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle\right). \tag{2}$$

Since this has to hold for all Z, this gives the unique possibility for $\nabla_X Y$.

To show existence define a quantity $\nabla_X Y$ by (2). To show this is a connection, let $f \in C^\infty(M)$ and calculate that $\langle\nabla_{fX} Y, Z\rangle$ equals

$$\begin{aligned}&\frac{1}{2}\left(fX\langle Y, Z\rangle + Y\langle Z, fX\rangle - Z\langle fX, Y\rangle - \langle Y, [fX, Z]\rangle - \langle Z, [Y, fX]\rangle + \langle fX, [Z, Y]\rangle\right)\\ =\ &\frac{1}{2}\left(fX\langle Y, Z\rangle + Y(f\langle Z, X\rangle) - Z(f\langle X, Y\rangle) - \langle Y, [fX, Z]\rangle - \langle Z, [Y, fX]\rangle + f\langle X, [Z, Y]\rangle\right)\end{aligned}$$

Now $[fX, Z] = fXZ - Z(fX) = f[X, Z] - Z(f)X$. Using this and the Liebniz rule for vector fields the above becomes $\langle f\nabla X_Y, Z\rangle$ and thus ∇ is linear over smooth functions in X.

To show the Liebniz rule for the connetion let h be a smooth function and calculate

$$\begin{aligned}\langle\nabla_X(hY), Z\rangle &= \frac{1}{2}\left(X(h\langle Y, Z\rangle) + hY\langle Z, X\rangle\right.\\ &\quad \left.- Z(h\langle X, Y\rangle) - h\langle Y, [X, Z]\rangle - \langle Z, [hY, X]\rangle + \langle X, [Z, hY]\rangle\right)\\ &= \frac{1}{2}\left(X(h)\langle Y, Z\rangle) + hX\langle Y, Z\rangle + hY\langle Z, X\rangle - Z(h)\langle X, Y\rangle - hZ\langle X, Y\rangle\right.\\ &- \left. h\langle Y, [X, Z]\rangle - h\langle Z, [Y, X]\rangle + X(h)\langle Z, Y\rangle + h\langle X, [Z, Y]\rangle + Z(h)\langle X, Y\rangle\right)\\ &= X(h)\langle Y, Z\rangle + h\langle\nabla_X(Y), Z\rangle\end{aligned}$$

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after some cancellation. Since this holds for all Z we deduce

$$\nabla_X(hY) = X(h)Y + h\nabla_X(Y).$$

But $X(h)$ is the contraction of dh with X, so we actually have $\nabla(hY) = dh \otimes Y + h\nabla(Y)$. Since it is clearly linear (over the reals) in Y this proves it is a connection.

Now, to show compatibility with g, we compute $\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ equals

$$\begin{aligned}&\frac{1}{2}\left(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle Y,[X,Z]\rangle - \langle Z,[Y,X]\rangle + \langle X,[Z,Y]\rangle\right)\\ +&\frac{1}{2}\left(X\langle Z,Y\rangle + Z\langle Y,X\rangle - Y\langle X,Z\rangle - \langle Z,[X,Y]\rangle - \langle Y,[Z,X]\rangle + \langle X,[Y,Z]\rangle\right)\end{aligned}$$

and all terms cancel except for $X\langle Y,Z\rangle$ and thus $\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = X\langle Y,Z\rangle = \nabla_X\langle Y,Z\rangle$ which proves it is compatible with g.

Finally to show ∇ is symmetric, we compute $\langle \nabla_X Y, Z\rangle - \langle \nabla_Y X, Z\rangle$ is equal to

$$\begin{aligned}&\frac{1}{2}\left(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle Y,[X,Z]\rangle - \langle Z,[Y,X]\rangle + \langle X,[Z,Y]\rangle\right)\\ -&\frac{1}{2}\left(Y\langle X,Z\rangle + X\langle Z,Y\rangle - Z\langle Y,X\rangle - \langle X,[Y,Z]\rangle - \langle Z,[X,Y]\rangle + \langle Y,[Z,X]\rangle\right)\end{aligned}$$

and all the terms cancel except for $-\frac{1}{2}\langle Z,[Y,X]\rangle + \frac{1}{2}\langle Z,[X,Y]\rangle = \langle [X,Y],Z\rangle$. Thus

$$\langle \nabla_X Y, Z\rangle - \langle \nabla_Y X, Z\rangle = \langle [X,Y],Z\rangle$$

for all Z and so $\nabla_X Y - \nabla_Y X = [X,Y]$ which says exactly that ∇ is symmetric. □

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