BOUNDARY ORIENTATION

My account of the definition of the orientation on the boundary had a gap that I would like to fix. I proved the following:

Lemma 1. *Suppose that N is a manifold and for each p we have chosen an orientation μ_p for T_pN. Suppose that N is covered by open sets U that each admit an oriented frame. Then N is orientable.*

Now we wish to apply this to the case that $N = \partial M$ is the boundary of some manifold-with-boundary M. To do this let $p \in \partial M$, and pick a chart (U, ϕ) for p in M. Then $\phi(U)$ is an open set in $\mathbb{H}^n$ and $\phi(p) \in \partial \mathbb{H}^n$. Now on $\mathbb{H}^n$ we have vector fields

$$e_i = \frac{\partial}{\partial x_i} \quad \text{for } i = 1, \ldots, n-1$$

and

$$e_n = -\frac{\partial}{\partial x_n}$$

Note that e_n is outward pointing when restricted to $\partial \mathbb{H}^n$. Pull these back to M by setting

$$X_1 = D\phi^{-1} e_n$$
$$X_i = D\phi^{-1} e_{i-1} \quad \text{for } i = 2, \ldots, n.$$

Then $X_1, \ldots, X_n$ is a local frame for TM. If this frame is oriented then leave it alone, otherwise replace X_n with $-X_n$ to get an oriented frame. Now observe that $X_2|_{\partial M}, \ldots, X_n|_{\partial M}$ all lie in the tangent space to ∂M (this is what I missed out in lecture!) and thus give an oriented frame for $T\partial M$, and so we can apply the lemma.

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