Part III 2011: Differential geometry (Version 1: November 29, 2011)

Example Sheet 3

1. Show directly that if ω is a 1-form and $X, Y \in \mathrm{Vect}(M)$ then $X\omega(Y) - Y\omega(X) - \omega([X,Y])$ defines a 2-form. [*Hint: show it is linear over smooth function. The point is one can use this expression to given an invariant* definition *of $d\omega$, and an exercise from the previous sheet shows this agrees with the definition from lectures*]

2. Show how an orientation on a manifold M defines a nowhere vanishing n-form ω, and that two such forms ω_1 and ω_2 give the same orientation if and only if $\omega_1 = f\omega_2$ for some positive smooth function M. Deduce that if M is connected and orientable then it has precisely two orientations up to equivalence.

3. Given a connection d_E we have the curvature $\mathcal{R} = d_E \circ d_E$. We said in lecture that we think of this as an element in $\Omega^2(End(E)$ which for this question we shall denote by R. Show that, under the various identifications and abuse of notations involved, that
$$\mathcal{R}(\sigma) = R \wedge \sigma.$$
[*Extension:* Show also that for all $p \geq 0$, the map $d_E \circ d_E : \Omega^p(E) \to \Omega^{p+2}(E)$ is also given by $\sigma \mapsto R \wedge \sigma$]

4. Let d_E and d_F be connections on bundles E and F over the same anifold M. Define
$$d_{E\otimes F}(s \otimes s') = d_E s \otimes s' + s \otimes d_F s'$$
where s and s' are local sections of E and F respectively. Show that this defines a connection on $E \otimes F$.

5. (The Bianchi Identity) If d_E is a connection on a vector bundle E, let $d_{End(E)}$ denote the connection on $\mathrm{End}(E)$ whose covariant dervative $\nabla_X^{End(E)}$ is defined by
$$\nabla_X^E(\theta(\sigma)) = \nabla_X^{End(E)}\theta(\sigma) + \theta(\nabla_X^E \sigma)$$
for any section θ of $\mathrm{End}(E)$, section σ of E and vector field X. Set d^{Hom} to denote the corresponding covariant derivative. With $R \in \Omega^2(\mathrm{End}(E))$ denoting the curvature of E, prove that $d_{\mathrm{End(E)}}(R) = 0$.

6. Define what it means for g to be a metric on a vector bundle E, and prove that if g and g' are metrics on E and F then there are induced metrics on E^* and $E \otimes F$.

   Now suppose (M, g) is a Riemannian manifold and ∇ is a connection on M. Show that if ∇ is compatible with g then the induced connection the tensor bundle $\mathcal{T}^{(k,l)}M$ is compatible with the induced metric on this bundle.

7. (This exercise uses the summation convention) Let g be a Riemannian metric on M. Suppose we have local coordinates $x_1, \ldots, x_n$ and write $g = g_{ij}dx_i \otimes dx_j$ for smooth functions g_{ij}. If ∇ is a connection on M we defined the Christoffel symbols by
$$\nabla_{\frac{\partial}{\partial x_i}}\left(\frac{\partial}{\partial x_j}\right) = \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

Show that ∇ is symmetric if and only if $\Gamma^k_{ij} = \Gamma^k_{ji}$ for all i, j, k. Show also that ∇ defined the Levi-Civita connection if and only if

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right)$$

where g^{uv} is the inverse of g_{ij} (so $g_{ij} g^{jv} = \delta iv$).

8. Let M be an embedded submanifold of a manifold N. Show that there is a vector bundle $TN|_M$ on M, containing TM as a sub-bundle, and that any connection ∇ on N induces a connection (which we also denote as ∇) on $TN|_M$.

   Suppose now that g is a Riemannian metric on N; show that g induces a Riemannian metric on M, and also determines an orthogonal projection map $\pi : TN|_M \to TM$ of bundles on M. If now ∇ denotes the Levi-Civita connection on N, identify the Levi-Civita connection on M in terms of ∇ and π.

9. Suppose that g is a Riemannian metric on a smooth manifold M, and r is a strictly positive real number. Show that the effect of scaling the Riemannian metric by r^2 (and hence distances by r) is to scale the sectional curvatures by $1/r^2$.

10. Consider the embedded submanifold $S^{n-1} \subset \mathbb{R}^n$, the unit sphere, and let the symbol dS^2 denote the expression for the induced Riemannian metric on the unit sphere S^{n-1} with respect to suitable local coordinates on the sphere. Show that the Euclidean metric on $\mathbb{R}^n \setminus \{0\}$ can be expressed as $g = dr^2 + r^2 dS^2$, where $r = |x|$, $x \in \mathbb{R}^n$. [You might like to consider the dimensions $n = 2$ or 3 first, using polar coordinates.]

11. Suppose that $\mathbf{r} = \mathbf{r}(u, v)$ is a regular parameterization of a surface S in the affine space $\mathbb{R}^3$. There is a standard choice of a 'moving frame' (a basis of the tangent space $T_{\mathbf{r}}\mathbb{R}^3$) $\mathbf{r}_u$, $\mathbf{r}_v$, $\mathbf{n}$ at every point $\mathbf{r}$ of S, where $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v / |\mathbf{r}_u \times \mathbf{r}_v|$ is a unit normal vector to S. (Here the subscripts u and v at $\mathbf{r}$ are used to denote the respective partial derivatives.) Then there is a unique way to write the second derivatives of $\mathbf{r}$ as

$$\begin{aligned} \mathbf{r}_{uu} &= \Gamma^1_{11} \mathbf{r}_u + \Gamma^2_{11} \mathbf{r}_v + L\mathbf{n} \\ \mathbf{r}_{uv} &= \Gamma^1_{12} \mathbf{r}_u + \Gamma^2_{12} \mathbf{r}_v + M\mathbf{n} \\ \mathbf{r}_{vv} &= \Gamma^1_{22} \mathbf{r}_u + \Gamma^2_{22} \mathbf{r}_v + N\mathbf{n}, \end{aligned}$$

   for some functions Γ^i_{jk}, L, M, N on S. By deducing the expressions for Γ^i_{jk} in terms of the first fundamental form of S (i.e. the expression $Edu^2 + 2F dudv + Gdv^2$ for the metric in terms of the coordinates u, v), or otherwise, show that the Γ^i_{jk} are the Christoffel symbols for the Levi-Civita connection of the metric induced on S by restriction from the ambient $\mathbb{R}^3$. Use this to calculate the sectional curvature of the unit sphere.

12. Suppose we have two Riemannian manifolds (M, g) and (N, h); show that there is a natural product metric $g + h$ on $M \times N$. If X is a vector field on M and Y one on N,

we may regard both of these as vector field on the product. If ∇ denotes the Levi-Civita connection on $M \times N$, show that $\nabla_X Y = 0$. Conclude that $R(X, Y, X, Y) = 0$. [This means that product metrics have many sectional curvatures which are zero.]

13. Define hyperbolic space $\mathbb{H}$ to be the upper half plane $\{(x, y) : y > 0\}$ and set

$$g = \frac{dx \otimes dx + dy \otimes dy}{y^2}.$$

Verify that g defines a Riemannian metric on $\mathbb{H}$, and compute the Christoffel symbols of the Levi-Civita connecion. Show that $\gamma = (u, v)$ is a geodesic if and only if $u''v = 2u'v'$ and $v''v = v'^2 - u'^2$ and use this to find the geodesics.

Finally compute the Curvature tensor of $\mathbb{H}$ and show that the sectional curvature (i.e. the Gaussian curvature since this is dimension 2) is identially -1.

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