# DYNAMICAL SYSTEMS

Examples Sheet — Lent Term 2002

This is the only Examples Sheet for this course; it provides enough material for at least three supervisions.

Starred questions ask for (usually slightly pathological) counter-examples. There is no systematic way of doing such questions: if you don't see the trick within five minutes, go on to the next question.

1. Investigate the behaviour of $F^r$ for $x$ near 0 in each of the following cases of maps $F : \mathbf{R} \to \mathbf{R}$:

$$x \mapsto -x, \quad x \mapsto x + x^2, \quad x \mapsto x - x^3, \quad x \mapsto x + x^3.$$

2. Show that a fixed point cannot be both an attractor and a repellor. Can it be neither?

3*. Construct a piecewise continuous map $F : \mathbf{R} \to \mathbf{R}$ for which 0 is a fixed point with $F'(0) > 1$, but $F^r(x) \to 0$ as $r \to \infty$ for each $x$. [It is because of examples like this that the definition of an attractor $Q$ is not simply $F^r(P) \to Q$ as $r \to \infty$ for all $P$ near enough to $Q$.] By considering the behaviour of $F(x) - x$, show that there is no continuous map with this property.

4*. If $\mathcal{S}$ is the circle $\mathbf{R}/\mathbf{Z}$ and $Q$ a given point on it, construct a continuous map $F : \mathcal{S} \to \mathcal{S}$ such that $x_r \to Q$ for every $x_0$ but $Q$ is not an attractor.

5*. If $\mathcal{S}$ is either a closed interval or a circle, construct a piecewise continuous map $F : \mathcal{S} \to \mathcal{S}$ and a cycle $\{Q_1, Q_2\}$ of order 2 such that for every $x_0$ in $\mathcal{S}$ the omega-set of $x_0$ is contained in $\{Q_1, Q_2\}$. In each case, is there a continuous map with this property?

6. Let $F, G$ be $C^1$ maps $\mathbf{R} \to \mathbf{R}$ which are $C^1$ conjugate by $\phi$, let $x$ be a fixed point of $F$ and write $y = \phi(x)$ so that $y$ is a fixed point of $G$. Prove that $F'(x) = G'(y)$.

7. Suppose that $a, b$ are positive, and define maps of the real axis to itself by

$$F_a(x) = ax, \quad F_b(x) = bx.$$

Find necessary and sufficient conditions for $F_a$ and $F_b$ to be
(i) topologically conjugate,

(ii) $C^1$ conjugate.
Does the answer change if the maps are restricted to $[0, \infty)$, or to $(0, \infty)$?

Questions 8 to 14 relate to the map $F(x) = cx(1 - x)$ of $[0, 1]$ to itself, where $0 \leq c \leq 4$.

8. Use a computer to examine the limiting behaviour of $x_r$ as $r \to \infty$ for various values of $x$ (including $x = \frac{1}{2}$) and each of the following values of $c$:

$$2.6,\ 3.1,\ 3.5,\ 3.55,\ 3.566,\ 3.829,\ 3.845,\ 3.9222.$$

In each case decide whether or not the omega-set appears to be a cycle of finite order.

9. Sketch the graphs of $F$ and $F^2$ and find their fixed points, determining which of them are attractors.
[The answers will vary with $c$. Use the facts that $F^2$ has at most three extrema, one of which is $\frac{1}{2}$ by symmetry, and split cases on whether $\frac{1}{2}$ is a maximum or a minimum.]

10. By considering the graph of $F^2$, or otherwise, show that as $c$ increases through 3 a cycle of period 2 is created, which is initially an attractor.

11. Suppose that $3 < c < 1 + \sqrt{6}$.
(i) For each $r > 1$, show that there are just two points $x$ such that

$$F^{r+1}(x) = (c - 1)/c, \quad F^r(x) \neq (c - 1)/c,$$

and that the set of all such points as $r$ varies has 0 and 1 as its only points of accumulation.
(ii) Show that there is just one 2-cycle and that it is an attractor.
(iii) If $x$ is not such that $F^r(x) = 0$ or $(c - 1)/c$ for some $r$, show that $x$ is in the domain of attraction of the 2-cycle.

12. Suppose that $1 < c \leq 2$ and write $y = (c - 1)/c$, so that $y$ is a fixed point of $F$. Prove
(i) if $0 < x_0 < y$ then $x_r$ is monotone increasing as $r \to \infty$ and $x_r \to y$;
(ii) if $y < x_0 < 1$ then either some $x_r \leq y$ or $x_r$ is monotone decreasing as $r \to \infty$.
Deduce that $x_r \to y$ for all $x_0$ in $(0, 1)$.

13. Suppose that $2 < c \leq 3$ and write $y = (c - 1)/c$, so that $y$ is a fixed point of $F$. Prove

(i) $F^2$ is a map of $[\frac{1}{2}, y)$ into itself, and $F^2(x) > x$ there;
(ii) $F$ maps $[c^{-1}, \frac{1}{2}]$ into $[\frac{1}{2}, y]$;
(iii) if $0 < x < c^{-1}$ then the iterates $x_r$ are monotone increasing as long as they stay in $(0, c^{-1})$, and the first iterate not in this interval lies in $[c^{-1}, y)$;
(iv) if $y < x < 1$ then $0 < F(x) < y$.
Deduce that $x_r \to y$ for all $x_0$ in $(0, 1)$.
[The argument sketched in the last two questions is the traditional low-brow way of proving that the domain of attraction of $y$ is $(0, 1)$. Both it and the argument given in the lectures show that while deciding whether a point is an attractor is usually straightforward, finding its domain of attraction is not.]

14. (i) Show that the points of period 3 correspond to the roots of a sextic equation. For $c = 4$ use the transformation $x = \sin^2(\frac{1}{2}\pi\theta)$ to find them explicitly. Hence deduce that there is an interval $c_0 < c \leq 4$ in which they are real and distinct.
(ii) Use a computer to estimate $c_0$ and to discover what happens at $c = c_0$.

15. Let $a, b$ be such that $0 < a < b < 1$, and let $F_a, F_b$ be the maps $x \mapsto ax(1-x)$ and $x \mapsto bx(1-x)$ on $[0, 1]$.
(i) Suppose that $F_a, F_b$ are topologically conjugate through the homeomorphism $\phi$ of $[0, 1]$ to itself. Prove that $\phi(\frac{1}{2}) = \frac{1}{2}$ and deduce that $\phi$ maps $\frac{1}{4}a$ to $\frac{1}{4}b$. Hence show that any orientation-preserving homeomorphism from $[\frac{1}{4}a, \frac{1}{2}]$ to $[\frac{1}{4}b, \frac{1}{2}]$ can be extended in just one way to such a topological conjugacy $\phi$.
(ii) Making use of crude estimates for $F_a^r(\frac{1}{2})$ and $F_b^r(\frac{1}{2})$, show that $\phi$ cannot be differentiable at 0; and deduce that $F_a$ and $F_b$ are not $C^1$ conjugate.

16. Let $1 < a < 3$ with $a \neq 2$, and let $F_a, F_2$ be respectively the maps $x \mapsto ax(1-x)$ and $x \mapsto 2x(1-x)$ from $(c, 1-c)$ to itself, where $c > 0$ is so small that these maps are well defined. Show that $F_a, F_2$ are not topologically conjugate and deduce that $F_2$ is not structurally stable.

17. Show that the map $F(x) = \frac{1}{3}(x+1)$ on [0,1] is structurally stable.

18. Let $f$ be a $C^1$ orientation-preserving circle map with a lift $F$ such that $\rho(F) = p/q$ is rational, and assume that $f$ has only finitely many points of period $q$. Show that $f$ is structurally stable if and only if the graphs of $y = F^q(x)$ and $y = x + p$ do not touch.

19. Let $F : \mathbf{R} \to \mathbf{R}$ be given by $F(x) = x + c$. Show that $F$ is a lift of a circle map $f$, and that

**(i)** if $c$ is rational then every point of the circle is periodic;

**(ii)** if $c$ is irrational then $f$ has no periodic points;

**(iii)** if $c$ is irrational then $f$ is transitive and topologically transitive;

**(iv)** $f$ does not have sensitive dependence on initial conditions;

**(v)** as $c$ varies the family of maps $f$ is not structurally stable.

Show that $f$ has degree 1. What is its rotation number?

20. Let $F : \mathbf{R} \to \mathbf{R}$ be given by $F(x) = x + c + b \sin 2\pi x$. Show that $f$ is a lift of a circle map $f$ of degree 1 and that $f$ is orientation preserving if $|2\pi b| < 1$. If also $|c| < |b|$ show that $\rho(f) = 0$.

21. Show that if $c_1, c_2$ are strictly between 0 and $1/2\pi$ then

$$F_1(x) = x + c_1 \sin^2 2\pi x \quad \text{and} \quad F_2(x) = x - c_2 \cos^2 2\pi x$$

are lifts of circle maps with the same rational rotation number, even though $F_1(x) > F_2(x) > F_1(x) - 1$ for all $x$.

22. Let $f_1, f_2$ be orientation preserving circle maps of degree 1 with lifts $F_1, F_2$ such that $F_1(x) > F_2(x) > F_1(x) - 1$. If $f_1$ is twice continuously differentiable and has irrational rotation number $\rho(f_1)$, prove that $\rho(f_2) \neq \rho(f_1)$. [Use Denjoy's theorem.]

23. [Arnold tongues] Show that if $\delta, \epsilon$ are both small the map

$$F(x) = x + \tfrac{1}{2} + \delta + \epsilon \sin 2\pi x$$

is a lift of an orientation preserving circle map of degree 1. Show that the region in $(\delta, \epsilon)$ space in which $\rho(F) = \frac{1}{2}$ has the form

$$-\tfrac{1}{2}\pi\epsilon^2 + O(\epsilon^3) \leq \delta \leq \tfrac{1}{2}\pi\epsilon^2 + O(\epsilon^3).$$

24. [Devil's staircase] Show that for any $c$ the map $\mathbf{R} \to \mathbf{R}$ given by

$$F_c(x) = x + c + (\sin 2\pi x)/4\pi$$

is a lift of an orientation preserving circle map of degree 1. For any integers $p \geq 0$ and $q > 0$ show that $F_c^q(x) - p$ is not the identity map $x \mapsto x$. Show that we can choose $c_1, c_2$ so that

$$F_{c_1}^q(x) < x + p < F_{c_2}^q(x)$$

for all $x$, and deduce that there exists an interval of values of $c$ for which $F_c^q(x) = x + p$ has a real root. Hence show that there is an interval of values of $c$ for which $\rho(F_c) = p/q$.

25. Let $f$ be a continuous map of $[0, 1]$ to itself and let $x_0, x_1, x_2, x_3$ be points in $[0, 1]$ with $x_0 < x_1 < x_2 < x_3$. Suppose that $f(x_n) = x_{n+1}$ where subscripts are to be taken modulo 4. Show that $f$ has a periodic orbit of each exact period.

26. If $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ show that $A^{n+2} = A^{n+1} + A^n$. Show that $A^5$ has trace 11 and deduce that the subshift map defined by $A$ has just two cycles of exact period 5. What are they?

Repeat the calculation for cycles of exact period 6.

27. Consider the subshift $\Sigma_W$ on the two symbols 0,1 with forbidden words

$$W = \{000,\ 111,\ 010,\ 101\}.$$

Obtain the counting function. What are the permitted sequences?

28. There are 16 $2 \times 2$ matrices each of whose elements is 0 or 1. Which of these correspond to subshift maps with at least one admissible sequence? For each such, describe the set of all admissible sequences.

29*. Let $\sum a_n X^n$ and $\sum b_n X^n$ be counting functions for subshift maps, not necessarily of finite type. Prove that $\sum a_n b_n X^n$ and $\sum (a_n + b_n) X^n$ are also counting functions for subshift maps, and if the first two functions come from maps of finite type, so do the last two. Show also that if $\sum c_n X^n$ is a counting function for a subshift map then either $c_n = O(1)$ or there are arbitrarily large $n$ for which $c_n > n$, and that this last inequality cannot be improved.

30. With the alphabet $\mathcal{A} = 0, 1$ let the set of forbidden words be

$$\mathcal{W} = 0101, 001001, 00010001, \ldots.$$

Prove that $\mathcal{S}_{\mathcal{W}}$ is not a subshift space of finite type.

31*. Exhibit a subshift space of finite type such that the map $\sigma$ is not onto.

32. In 2 dimensions, find the index numbers of saddle points, nodes and foci, splitting cases as necessary. Suppose that the underlying map depends on a parameter $\lambda$; if two isolated fixed points (each being of one of these types) coincide when $\lambda = 0$, in what circumstances can they annihilate each other?

33. Let $f_\lambda$ be a map $\mathbf{R}^2 \to \mathbf{R}^2$ depending on a parameter $\lambda$. Suppose that $O$ is an isolated fixed point of $f_\lambda$ for $\lambda < 0$ which is a saddle point, and that $\{P_1, P_2\}$ is a cycle of order 2 such that $P_1$ and $P_2$ tend to $O$ as $\lambda \to 0$. Investigate the bifurcation at $O$ when $\lambda = 0$. What can you say about the behaviour of the eigenvalues of the saddle point?

34. [The adding machine] Let $f : I \to I$ with $I = [0, 1]$ be a continuous map. Define the map $f_1 = D(f)$, the *double* of $f$, by

$$f_1(x) = \begin{cases} \frac{2}{3} + \frac{1}{3}f(3x) & \text{for } 0 \leq x \leq \frac{1}{3}, \\ (2 + f(1))(\frac{2}{3} - x) & \text{for } \frac{1}{3} \leq x \leq \frac{2}{3}, \\ x - \frac{2}{3} & \text{for } \frac{2}{3} \leq x \leq 1. \end{cases}$$

Prove that $f_1$ has exactly one fixed point and it is repelling, that any other periodic orbit of $f_1$ has even order, and that there is a one-one correspondence between orbits of period $2n$ of $f_1$ and orbits of period $n$ of $f$. Prove also that corresponding orbits have the same stability.

Now let $f_0(x) = \frac{1}{3}$ for all $x$, and write $f_{n+1} = D(f_n)$ for all $n \geq 0$. Show that the periodic orbits of $f_n$ are one repelling orbit of period $2^r$ for each $r$ in $0 \leq r < n$ and one attracting orbit of period $2^n$.

Let $f_\infty(x) = \lim f_n(x)$ as $n \to \infty$. Prove that $f_\infty$ is continuous, that its only periodic orbits are one orbit of period $2^n$ for each $n$, and that all these are repelling. Show also that the points of this orbit lie one in each of the $2^n$ intervals which are removed at the $n$th stage of constructing the "middle third" Cantor set. If $\Lambda$ is the Cantor set, show that $\Lambda$ is $f_\infty$-invariant.

Let $\Sigma_2$ consist of the singly infinite sequences $\{a_0, a_1, \ldots\}$ where each $a_i$ is 0 or 1, and let $\phi : \Sigma_2 \to \Sigma_2$ be defined by adding 1 to $a_0$ and carrying forward as far as necessary — that is, add 1 to $\ldots a_2 a_1 a_0$ in the scale of 2. Prove that $f_\infty$ acting on $\Lambda$ is topologically conjugate to $\phi$ acting on $\Sigma_2$.

35. Let $\Sigma_2$ consist of the doubly infinite sequences $\{\ldots, a_{-1}, a_0, a_1, \ldots\}$ where each $a_i$ is 0 or 1, and let $\sigma$ be the shift map. For any $p$ in $\Sigma_2$ define the *stable set* $W^s(p)$ to consist of those sequences which agree with $p$ to the right of some place, and the *unstable set* $W^u(p)$ to consist of those which agree with $p$ to the left of some place. A point other than $p$ which lies in $W^s(p) \cap W^u(p)$ is called *homoclinic* at $p$. Prove that the points homoclinic to $p$ are dense in $\Sigma_2$.

36. Let $\phi$ be the Smale horseshoe map and $\Lambda$ the set of points none of whose images under any positive or negative power of $\phi$ lie outside the defining

square. Prove that $\phi$ restricted to $\Lambda$ is conjugate to $\sigma$ restricted to $\Sigma_2$ in the notation of the previous question.

37. Investigate the singular points of the Hénon map

$$(x, y) \mapsto (A - By - x^2, x),$$

splitting cases as necessary.

38. Investigate numerically the behaviour of the Hénon map when $A = 1 \cdot 4$ and $B = -0 \cdot 3$. [At these values the map appears to have a strange invariant set which is an attractor, though I do not believe that this has been rigorously proved.]