AFFINIZING QUANTUM ALGEBRAS: FROM D-MODULES TO K-THEORY

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There are many situations in representation theory where an algebra A_q is contructed in the Grothendieck group of certain mixed perverse sheaves on a variety X. We mention some:

i) Let G be an algebraic group, $\mathcal{B}$ its flag variety. Then G-equivariant perverse sheaves on $\mathcal{B} \times \mathcal{B}$ realise the Iwahori-Hecke algebra of G.

i') More generally, if K is the fixpoints of an involution $\theta : G \to G$, K-equivariant perverse sheaves on $\mathcal{B}$ realise the "coherent continuation" representation; an object of fundamental importance for the representation theory of real groups.

ii) For G as in (i), the "character sheaves" of Lusztig [L3] encode the character theory of the finite groups G^F, for F the Frobenius map for an F_q structure on G.

ii') More generally, one may consider the character sheaves on G/K, as in [G2].

iii) Certain perverse sheaves on the moduli space of representations of a quiver define $U_q\mathfrak{n}$, the Hopf algebra (quantum group) associated to the $-$ part of a Kac-Moody algebra.

In each of these situations, the fundamental object is a certain Lagrangian variety $\Lambda \subseteq T^*X$, and the category of perverse sheaves is precisely those whose characteristic variety lies in Λ. Further, Λ is "nilpotent" in some suitable sense, and contained in the inverse image of 0 under a moment map. (In fact, this complex symplectic structure seems to often come from an underlying hyper-Kähler structure; but I have no understanding of the representation theoretic meaning of this). Note that we can add the "geometric Langlands program" to the list of such examples.

The main observation of this paper, illustrated in detail in case (iii), is that to this data one can attach another object, the "quantum affinised algebra" $\widehat{A_q}$ with trivial central charge. In the case where there is a 1-1 correspondence between the irreducible components of Λ and the irreducible equivariant perverse sheaves with characteristic variety in Λ, this algebra is realised in $K^{G\times \mathbf{C}^*}(\Lambda)$, the equivariant K-homology of Λ. In other cases, such as (i'), K-theory is probably not the correct object, and perhaps one should use something like $\mathcal{E}$-modules. Alternately, it may be one should do this one block at a a time (see [Vo]).

In this geometric context, the evident algebra morphism $A_q \hookrightarrow \widehat{A_q}$ (which is clear on the level of generators and relations) becomes realised by the theorem of Gabber that the weight filtration on the nearby cycles coincides with the monodromy

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filtration. However, for technical reasons we work with M. Saito's mixed Hodge modules, and not with the mixed l-adic sheaves (see 1.7.1, below).

The significance of this in case (i) was realised by Tanisaki [T]. This paper can thus be considered as a commentary on [T] and the spectacular paper [KL].

In a sequel, in preparation, we intend to deduce the finite dimensional representation theory of $U_q\widehat{\mathfrak{g}}$ from these techniques and the additional insight provided by [N,11].

The second observation of this paper is that we may get rid of the Lagrangian variety altogethor, at the expense of modifying the multiplication on $K^{G\times\mathbf{C}^*}(X)$. In the case of $U_q\mathfrak{n}$ this means we can construct $U_q\widehat{\mathfrak{n}}$ inside the equivariant cohomology of a point.

To use the language of I. Frenkel, we have "categorified" a q-vertex operator algebra. This construction of the quantum affine algebra inside symmetric polynomials is not new; B. Feigin has also discovered this (unpublished). Here however it has the peculiar consequence that all the complexity of the Lagrangian variety Λ is encoded in certain polynomials (essentially the Chern classes of their normal bundles). Further, using the "characteristic variety" embedding $U_q\mathfrak{n} \hookrightarrow U_q\widehat{\mathfrak{n}}$, we can represent each element of Lusztig's canonical basis as an I-tuple of polynomials. Then properties of these basis elements (such as which representations they vanish in) become certain divisbility properties of these polynomials. We will return to this in a later paper.

Remark: In the original version of this work, I proved theorem (7.8) by explicit computation in K-theory for rank 2, combined with certain arguments paralleling those of [L1,L2]. This would have taken many more pages to write, and made the result seem quite miraculous. The present treatment arose from my various attempts to understand *why* the theorem is true. For an explicit proof by calculation, see [G1].

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1. Mixed Hodge modules

It is a remarkable theorem of M. Saito that every simple $\mathcal{D}$-module of geometric origin carries a canonical good filtration. As a consequence, given a simple $\mathcal{D}$-module, we obtain a $\mathbf{C}^*$-equivariant coherent sheaf supported on the characteristic variety of the $\mathcal{D}$-module. The following paragraphs recall the notations of filtered $\mathcal{D}$-modules in order to precisely state this fact, and to describe how they behave under morphisms which preserve semisimplicity, i.e. inverse image by a smooth map, direct image by a proper map, and restriction to the zero section of a monodromic sheaf on a vector bundle.

The definitions below are extracted from [La1] and [S1,S2]. In what follows, X will denote a smooth quasi-projective complex algebraic variety, $\mathcal{O}_X$ its structure sheaf, and $\pi : T^*X \to X$ the cotangent bundle to X.

1.1 Let $\mathcal{D}_X$ denote the $\mathcal{O}_X$-algebra of differential operators on X, equipped with the increasing filtration by order $\mathcal{D}_{X,i}$. A filtered $\mathcal{D}_X$-module (M,F) is a left $\mathcal{D}_X$-module M equipped with an increasing filtration F^i such that $\cup F^iM = M$, $F^iM = 0$ for $i << 0$, and $\mathcal{D}_{X,i}.F^jM \subseteq F^{j+1}M$. If F^iM is a quasi-coherent $\mathcal{O}_X$-module for each i, we say (M,F) is quasi-coherent, and we let $MF_{\mathrm{qcoh}}(\mathcal{D}_X)$ denote the category of quasi-coherent filtered $\mathcal{D}_X$-modules (morphisms respect filtrations). This is an exact category, and we let $DF_{\mathrm{qcoh}}(\mathcal{D}_X)$ denote its derived category [La1].

Let $M^{\mathbf{C}^*}_{\mathrm{qcoh}}(\mathcal{O}_X)$ (resp. $M^{\mathbf{C}^*}_{\mathrm{coh}}(\mathcal{O}_X)$) denote the category of graded quasi-coherent (resp. coherent) $\mathcal{O}_X$-modules, and $D^{\mathbf{C}^*}_{\mathrm{qcoh}}(\mathcal{O}_X)$ the derived category of $M^{\mathbf{C}^*}_{\mathrm{qcoh}}(\mathcal{O}_X)$, $D^{\mathbf{C}^*}_{\mathrm{perf}}(\mathcal{O}_X)$ the derived category of perfect complexes.

1.2 Given a filtered $\mathcal{D}_X$-module (M,F), we let $\mathrm{gr}_F M = \oplus F^i M/F^{i-1}M$, a graded $\mathrm{gr}\mathcal{D}_X$-module. As $\mathrm{gr}\mathcal{D}_X = \pi_*\mathcal{O}_{T^*X}$ and π is affine, we get a functor $\widetilde{\mathrm{gr}}$: $MF_{\mathrm{qcoh}}(\mathcal{D}_X) \to M^{\mathbf{C}^*}_{\mathrm{qcoh}}(\mathcal{O}_{T^*X})$ which lifts to a functor $DF_{\mathrm{qcoh}}(\mathcal{D}_X) \to D^{\mathbf{C}^*}_{\mathrm{qcoh}}(\mathcal{O}_{T^*X})$.

1.3 Recall F is a good filtration if $\mathrm{gr}_F M$ is coherent over $\mathrm{gr}\mathcal{D}_X$, and (M,F) is holonomic if F is good and $\dim\operatorname{supp}(\widetilde{\mathrm{gr}}M) = \dim X$. We say (M,F) is holonomic Cohen-Macauley if it is holonomic and $\mathrm{Ext}^i_{\mathrm{gr}\mathcal{D}_X}(\mathrm{gr}M, \mathrm{gr}\mathcal{D}_X) = 0$ for $i \neq \dim X$; and write $MF_{\mathrm{hcm}}(\mathcal{D}_X)$ for the full subcategory of holonomic Cohen-Macauley filtered $\mathcal{D}_X$-modules.

For (M,F) in $DF_{\mathrm{qcoh}}(\mathcal{D}_X)$, define $\mathbf{D}(M,F) = \mathrm{Hom}((M,F), \mathcal{D}_X \otimes \omega_X^{-1})[\dim X]$ where ω_X has the trivial filtration and the filtration induced on Hom is $F^i\mathrm{Hom}(X,Y) = \{\phi \in \mathrm{Hom}(X,Y) \mid \phi(F^a X) \subseteq F^{a+i}Y \text{ for all } i\}$. Then $\mathbf{D} : DF_{\mathrm{qcoh}}(\mathcal{D}_X) \to DF_{\mathrm{qcoh}}(\mathcal{D}_X)^{\circ}$, and $\mathbf{D}^2 = 1$. Also define $\mathbf{D} : D^{\mathbf{C}^*}_{\mathrm{qcoh}}(\mathcal{O}_{T^*X}) \to D^{\mathbf{C}^*}_{\mathrm{qcoh}}(\mathcal{O}_{T^*X})^{\circ}$ by $\mathbf{D}\mathcal{A} = \mathrm{RHom}(\mathcal{A}, \omega_{T^*X})[\dim X]$. If now $(M,F) \in MF_{\mathrm{hcm}}(\mathcal{D}_X)$, then $\mathbf{D}(M,F) \in MF_{\mathrm{hcm}}(\mathcal{D}_X)$ also; and $\widetilde{\mathrm{gr}}\circ\mathbf{D}(M,F) = \mathbf{D}\widetilde{\mathrm{gr}}(M,F)$.

1.4 Now let $f : X \to Y$ be a morphism of varieties. Then there are functors $f_!, f_* : DF_{\mathrm{qcoh}}(\mathcal{D}_X) \to DF_{\mathrm{qcoh}}(\mathcal{D}_Y)$, $f^!, f^* : DF_{\mathrm{qcoh}}(\mathcal{D}_Y) \to DF_{\mathrm{qcoh}}(\mathcal{D}_X)$ which on forgetting the filtration become the usual operations between $D(\mathcal{D}_X)$ and $D(\mathcal{D}_Y)$ [La1,Sa1]. We recall that, forgetting filtration $f_* = Rf_*(\mathcal{D}_{Y\leftarrow X} \otimes^L{}_{\mathcal{D}_X} \cdot)$, $f^* = \mathcal{D}_{X\to Y} \otimes^L{}_{f^{-1}\mathcal{D}_Y} f^{-1}(\cdot)$, and that $f_! = \mathbf{D}f_*\mathbf{D}$, $f^! = \mathbf{D}f^*\mathbf{D}$.

1.5 Write $MHM(X)$ for the abelian category of mixed Hodge modules of M. Saito; $D^bMHM(X)$ for its derived category. If $f : X \to Y$ is a morphism of varieties, there are functors $f^!, f^*, f_!, f_*$ between $D^bMHM(X)$ and $D^bMHM(Y)$. There is a forgetful functor $MHM(X) \to MF_{\mathrm{hcm}}(\mathcal{D}_X)$ which commutes with Verdier duality $\mathbf{D}$. If $f : X \to Y$ is smooth, then the forgetful functor composed with f^* is the functor $f^* : DF_{\mathrm{qcoh}}(Y) \to DF_{\mathrm{qcoh}}(X)$ of [La1] referred to above. If $f : X \to Y$ is proper, then the forgetful functor composed with $f_!$ is the functor $f_! : DF_{\mathrm{qcoh}}(X) \to DF_{\mathrm{qcoh}}(Y)$ of [La1] above.

Write $\mathcal{M}^m X$ for the full subcategory of $MHM(X)$ whose objects are of geometric origin, and $D\mathcal{M}^m X$ for the full subcategory of $D^bMHM(X)$ whose objects have all perverse cohomology mixed Hodge modules of geometric origin. We have defined $\widetilde{\mathrm{gr}} : \mathcal{M}^m X \to M^{\mathbf{C}^*}_{\mathrm{coh}}(\mathcal{O}_{T^*X})$.

1.6 Now suppose an algebraic group G acts on X. We write $MHM_G(X)$ for the category of mixed Hodge modules on the stack X/G [T,BB], and $D^m_G(X)$ for the "equivariant derived category". Strictly speaking, the latter hasn't been defined in this generality (it is not the derived category of $MHM_G(X)$). For what follows below, we only need $MHM_G(X)$, whose objects are pairs (A,ϕ), $A \in MHM(X)$ and ϕ an isomorphism $a^*A \to p^*A$ satisfying the usual compatibilities, where $a : G \times X \to X$ is the action map, and $p : G \times X \to X$ the projection. If one needs to calculate RHom in $MHM_G(X)$ one can use the approximation to $EG \times_G X$ described in [BL]. (See [GL], for example).

We thus have a map $\widetilde{\mathrm{gr}} : \mathcal{M}^m_G(X) \to M^{G\times\mathbf{C}^*}_{\mathrm{coh}}(\mathcal{O}_{T^*X})$, where $\mathcal{M}^m_G(X)$ denotes the full subcategory of $MHM_G(X)$ whose objects (A,ϕ) have A of geometric origin, and $M^{G\times\mathbf{C}^*}_{\mathrm{coh}}(\mathcal{O}_{T^*X})$ denotes the $G \times \mathbf{C}^*$-equivariant coherent sheaves on T^*X

Warning: G, $\mathbf{C}^*$ play different roles here!

1.7 As a final piece of notation, suppose we are given (X,Λ,G), where $\Lambda\subseteq T^*X$ is a $\mathbf{C}^*$-stable Lagrangian subvariety, G acts on X and preserves Λ. Write $\mathcal{M}^m_G(X,\Lambda)$ for the full subcategory of $\mathcal{M}^m_G(X)$ consisting of objects for which the support of $\widetilde{\mathrm{gr}}$ is contained in Λ, and $K^{G\times\mathbf{C}^*}(\Lambda)$ for the Grothendieck group of $G\times\mathbf{C}^*$-equivariant sheaves on Λ, i.e. the Grothendieck group of compexes in $D^{G\times\mathbf{C}^*}_{\mathrm{perf}}(\mathcal{O}_{T^*X})$ with cohomology supported in Λ. As the map $i:\Lambda\hookrightarrow T^*X$ induces an isomorphism between $K^{G\times\mathbf{C}^*}(\Lambda)$ and the Grothendieck group of $G\times\mathbf{C}^*$-equivariant coherent sheaves on T^*X supported on Λ, we get a map, still denoted

$$\widetilde{\mathrm{gr}}:\mathcal{M}^m_G(X,\Lambda)\to K^{G\times\mathbf{C}^*}(\Lambda). \tag{1.7.1}$$

(Actually, as $\widetilde{\mathrm{gr}}(M,F)$ is reduced when M is regular, we already have $\widetilde{\mathrm{gr}}(M,F)=i_*A'$ for some coherent sheaf A' on Λ, without having to pass to the Grothendieck group).

2. Correspondences

Let $X\xleftarrow{f}Z\xrightarrow{g}Y$ be a diagram of smooth varieties such that the induced map $Z\to X\times Y$ is a closed embedding. This induces a functor $g_!f^*:D\mathcal{M}^m(X)\to D\mathcal{M}^m(Y)$. We call such a $X\xleftarrow{f}Z\xrightarrow{g}Y$ a correspondence between X and Y. This gives rise to a correspondence between T^*X and T^*Y, $T^*X\xleftarrow{q_f}T^*_Z(X\times Y)\xrightarrow{q_g}T^*Y$, where $T^*_Z(X\times Y)$ denotes the conormal bundle to Z in $X\times Y$, and the maps are as follows

$$\begin{tikzcd}
T^*_Z(X\times Y) \arrow[r] \arrow[d] & T^*Y\times_Y Z \arrow[r, "\pi_g"] \arrow[d, "{}^tg"'] & T^*Y \\
T^*X\times_X Z \arrow[r, "{}^tf"] \arrow[d, "\pi_f"'] & T^*Z & \\
T^*X & &
\end{tikzcd}$$

Let $\omega_{Z/Y}=g^{-1}(\omega_Y^{\otimes^{-1}})\otimes_{g^{-1}\mathcal{O}_Y}\omega_Z$, and $\rho:T^*_Z(X\times Y)\to Z$ be the evident map. We have a functor $I_{f,g}:D^{\mathbf{C}^*}_{\mathrm{qcoh}}(\mathcal{O}_{T^*X})\to D^{\mathbf{C}^*}_{\mathrm{qcoh}}(\mathcal{O}_{T^*Y})$ defined by

$$I_{f,g}A={q_g}_*(q_f^*A\otimes^L\rho^{-1}\omega_{Z/Y})[\dim Z-\dim X]. \tag{2.1.1}$$

Then the fundamental result, due in this form to Laumon [La1,La2], states that

$$\widetilde{\mathrm{gr}}\circ g_!f^*=I_{f,g}\circ\widetilde{\mathrm{gr}}:DF_{\mathrm{qcoh}}(\mathcal{D}_X)\to D^{\mathbf{C}^*}_{\mathrm{qcoh}}(\mathcal{O}_{T^*Y}). \tag{2.1.2}$$

One obtains this by base change from the two correspondences associated to f, g; the point is that the cartesian square above has no excess intersection bundle (it is "Tor independant").

Now suppose that f is smooth and g is proper. In this case, q_f is a regular morphism and q_g is proper. Then for $(M,F)\in MF_{\mathrm{hcm}}(X)$ we have that $I_{f,g}(M,F)\in D^{\mathbf{C}^*}_{\mathrm{perf}}(\mathcal{O}_{T^*Y})$. (More generally, one need only assume that both ${}^tf:\pi_f^{-1}(\operatorname{supp}\widetilde{\mathrm{gr}}M)\to T^*Z$ and $\pi_g:{}^tg^{-1}(\Gamma)\to T^*Y$ are proper maps, where $\Gamma={}^tf(\pi_f^{-1}(\operatorname{supp}\widetilde{\mathrm{gr}}M))$. We will not need this generality.)

If now $X \xleftarrow{f} Z \xrightarrow{g} Y$ is a correspondence and an algebraic group G acts equivariantly, and $\Lambda_X \subseteq T^*X$, $\Lambda_Y \subseteq T^*Y$ are as in (1.7) with $q_g(q_f^{-1}(\Lambda_X)) \subseteq \Lambda_Y$, then for f smooth and g proper we obtain a map,

$$I_{f,g} : K^{G\times \mathbf{C}^*}(\Lambda_X) \to K^{G\times \mathbf{C}^*}(\Lambda_Y). \tag{2.1.3}$$

Note that though Λ_X, Λ_Y may be singular, the morphisms used to define $I_{f,g}$ involve only pushforward by proper maps, and pullbacks by regular morphisms on the ambient smooth spaces $T^*X, T^*_Z(X\times Y), T^*Y$.

3. Inducing G-spaces

Suppose H is a subgroup of G, X an H-space, and let $G\times_H X$ be the induced space $(G\times X)/H$ where $h.(g,x) = (gh^{-1}, hx)$. Then the map $i : X \to G\times_H X$, $x \mapsto (1,x)$ induces an equivalence of categories $i^* : \mathcal{M}^m_G(G\times_H X) \to \mathcal{M}^m_H(X)$.

Write the induced correspondence on the cotangent bundle $T^*X \xleftarrow{\beta} X\times_{(G\times_H X)} T^*(G\times_H X) \xrightarrow{\alpha} T^*(G\times_H X)$. The natural map $G\times_H (X\times_{(G\times_H X)} T^*(G\times_H X)) \xrightarrow{\delta} T^*(G\times_H X)$ is an isomorphism, so α induces an isomorphism $\alpha^* : M^{G\times\mathbf{C}^*}_{\text{coh}}(T^*(G\times_H X)) \to M^{H\times\mathbf{C}^*}_{\text{coh}}(X\times_{(G\times_H X)} T^*(G\times_H X))$. Write $\mu_G : T^*(G\times_H X) \to \mathfrak{g}^*$, $\mu_H : T^*X \to \mathfrak{h}^*$ for the moment maps associated to the G-action (resp. H action). Then it is easy to check that if Y denotes the intersection of $X\times_{(G\times_H X)} T^*(G\times_H X)$ with $\mu_G^{-1}(0)$, then $\beta : Y \to T^*X$ induces an isomorphism $\beta : Y \to \mu_H^{-1}(0)$. Write $\gamma : \mu_H^{-1}(0) \to Y$ for its inverse.

We thus obtain an equivalence of categories $I_i = \beta_*\alpha^*[\dim G/H]$ from the derived category of $G\times\mathbf{C}^*$-equivariant perfect complexes on $T^*(G\times_H X)$ with cohomology supported in $\mu_G^{-1}(0)$ to that of $H\times\mathbf{C}^*$-equivariant perfect complexes with cohomology supported in $\mu_H^{-1}(0)$. Its inverse is given by $(\alpha^*)^{-1}\gamma_*$ and we have $\widetilde{\text{gr}}\circ i^* = I_i \circ \widetilde{\text{gr}}$, by (2.1.2).

More generally, given (X, Λ_X, H) define $\Lambda_{G\times_H X} \subseteq T^*(G\times_H X)$ by the image under δ of $G\times_H \gamma(\Lambda_X)$ (note $\Lambda_X \subseteq \mu_H^{-1}(0)$). Then we have described isomorphisms between $K^{G\times\mathbf{C}^*}(\Lambda_{G\times_H X})$ and $K^{H\times\mathbf{C}^*}(\Lambda_X)$.

4. Monodromic Sheaves

Let $p : E \to X$ be a vector bundle on a smooth variety X, p the projection and $i : X \to E$ the zero section. We have a natural action of $\mathbf{C}^*$ on E by homotheties; recall that a complex $A \in D\mathcal{M}^m(E)$ is said to be "monodromic" if t_a^*A is isomorphic to A for all $a\in\mathbf{C}^*$, here $t_a : E \to E$ is the multiplication by a map [V].

Then it is well known that for A monodromic, $p_*A = i^*A$ and $p_!A = i^!A$; so $p_*, p_!$ preserve weights and hence if A is pure (and therefore semisimple) so are $p_!A, p_*A$.

We now describe a functor i^*_m from certain coherent sheaves on T^*E supported on conic Lagrangian subvarieties to complexes of such sheaves on T^*X, with the property that $i^*_m\widetilde{\text{gr}}A = \widetilde{\text{gr}}(i^*A)$ for A a monodromic mixed Hodge module on E.

4.1 Let Λ be a conic Lagrangian subvariety of T^*E such that Λ is stable by the $\mathbf{C}^*$ action on T^*E induced by the action on E. This action induces a decomposition of $\mathcal{O}_{T^*E}$ into eigenspaces of the $\mathbf{C}^*$ action; and we say a coherent sheaf M on T^*E is "monodromic" if it has a decomposition $M = \oplus_{\lambda\in\mathbf{C}} M_\lambda$ compatible with the decomposition of $\mathcal{O}_{T^*E}$, i.e. $(\mathcal{O}_{T^*E})_n M_\lambda \subseteq M_{\lambda+n}$. For example, if $A\in MHM(X)$ is monodromic, $\widetilde{\text{gr}}A$ is monodromic in this sense.

Now let $j : T^*X \times_X E \hookrightarrow T^*E$ be the inclusion of the zero section, and denote by j^*M the formal linear combination of coherent sheaves on $T^*X \times_X E$ obtained in the usual way. (Namely, take the alternating sum of the cohomology of the complex $\mathcal{F}^{\cdot} \otimes_{\mathcal{O}_{T^*E}} M$, where $\mathcal{F}^{\cdot}$ is the Koszul complex, or any $\mathbf{C}^*$-equivariant resolution of $j_*\mathcal{O}_{T^*X\times_X E}$ by free $\mathcal{O}_{T^*E}$-modules. This virtual module is supported on $T^*X \times_X E$; filtering it by powers of the ideal in $\mathcal{O}_{T^*E}$ defining $T^*X \times_X E$ and taking the associated graded one gets a (virtual) coherent $\mathcal{O}_{T^*X\times_X E}$-module). Now j^*M admits a decomposition $j^*M = \oplus_{\lambda\in\mathbf{C}}(j^*M)_\lambda$ compatible with the $\mathbf{C}^*$ action on $T^*X \times_X E$, and each $(j^*M)_\lambda$ is a formal linear combination of coherent $\mathcal{O}_{T^*X}$-modules (this latter because for a finite dimensional vector space V, the symmetric power S^nV is still finite dimensional). Define $i_m^*M = \oplus_\lambda (j^*M)_\lambda$. We have

Proposition 4.2. *i_m^*M is a (virtual) coherent sheaf; i.e. $(j^*M)_\lambda = 0$ for all but finitely many λ.*

As i_m^* is clearly exact, we get in particular a well defined map

$$i_m^* : K^{\mathbf{C}^*}(\Lambda) \to K^{\mathbf{C}^*}(\Lambda_X)$$

where Λ_X is the image in T^*X of $j^{-1}(\Lambda)$ (more generally, we have defined i_m^* for any element of the Grothendieck group of monodromic sheaves on Λ; but we will only need $\mathbf{C}^*$-equivariant sheaves below). In fact, if a group G acts on E, and $L = Z_G(\mathbf{C}^*)$, we have

$$i_m^* : K^G(\Lambda) \to K^L(\Lambda_X).$$

Define also $p_!^m = i_m^! = \mathbf{D} i_m^* \mathbf{D}$. Note that 'morally' $p_!^m = I_{Id,p}$, $i_m^* = I_{i,Id}$ (in the notation of §2), and all we have done is explain how to cancel infinities (so, for example $\mathbf{C}[x] - x\mathbf{C}[x] = \mathbf{C}$).

Lemma 4.3. *If $F \to E$, $E \to X$ are two vector bundles, with zero sections j, i respectively, then $(ij)_m^* = i_m^* j_m^*$.*

Proof. This follows in a standard way from the fact that if a homomorphism of double complexes induces a quasi-isomorphism on each "vertical" single complex, then it induces a quasi-isomorphism on total complexes. The details are left to the reader.

Proposition 4.4. *Let $A \in MHM(E)$ be monodromic. Then $\widetilde{gr}(i^*A) = i_m^*\widetilde{gr}(A)$.*

Proof. As everything is local on X, we can assume $E = X \times \mathbf{C}^n$. Let (M, F) be the filtered $\mathcal{D}_E$-module obtained from A, θ the Euler vector field determined by the $\mathbf{C}^*$-action on E. As A is monodromic, $M = \oplus_{\lambda\in\mathbf{C}} M_\lambda$ where $M_\lambda = \{m \in M \mid (\theta - \lambda)^k m = 0, k >> 0\}$. Further, the filtration F is compatible with this weight space decomposition (i.e. $\theta F^p \subseteq F^p$). This last claim is part of the basic inductive definition of the mixed Hodge modules—that they are compatible with the vanishing cycles functors—and may be found in [S1,S2].

Now p_*M is quasi-isomorphic to the relative DeRham complex $\Omega^{\cdot}_{E/X} \otimes_{\mathcal{O}_E} M$. By the Cartan homotopy lemma applied to contraction with θ, each complex $(\Omega^{\cdot}_{E/X} \otimes_{\mathcal{O}_E} M)_\lambda = \Omega^{\cdot}_{E/X} \otimes_{\mathcal{O}_E} M_{\lambda-\cdot}$ is exact if $\lambda \neq 0$.

Then p_*A is represented by the complex $\Omega^{\cdot}_{E/X} \otimes_{\mathcal{O}_E} M_{0-\cdot}$ with filtration $F^p = \Omega^0_{E/X} \otimes_{\mathcal{O}_E} (F^pM)_0 \to \cdots \to \Omega^n_{E/X} \otimes_{\mathcal{O}_E} (F^{p+n}M)_{-n}$ (and other data). Hence the associated graded of this complex is just the Koszul complex calculating the restriction j as in (4.1); and so $\widetilde{\mathrm{gr}}(p_*A) = j^*(\widetilde{\mathrm{gr}}A)_0$. Also for $\lambda \neq 0$, $j^*(\widetilde{\mathrm{gr}}A)_\lambda$ is zero (when considered as a virtual $\mathcal{O}_{T^*X}$-module). Hence $\widetilde{\mathrm{gr}}(p_*A) = i_m^*\widetilde{\mathrm{gr}}A$ as claimed.

4.5 Let $p : E \to X$ be a vector bundle as above, and let $\check{p} : \check{E} \to X$ be the dual vector bundle, $ev : E \times_X \check{E} \to \mathbf{C}$ the natural pairing.

Define $\mathcal{F} : D^bMHM(E) \to D^bMHM(\check{E})$ by $\mathcal{F}(a) = \check{p}_!\phi_{ev}^{un}p^*(a)$, where ϕ_f^{un} is the unipotent part of vanishing cycles [F]. Then when restricted to $\mathcal{F} : D\mathcal{M}^m_{\mathbf{C}^*}(E) \to D\mathcal{M}^m_{\mathbf{C}^*}(\check{E})$ (i.e. monodromic sheaves on E with trivial monodromy eigenvalue) $\mathcal{F}$ is Fourier transform, and has the following properties:

i) $\mathcal{F}$ preserves weights. ii) If d is the rank of E, and $a \in \mathcal{M}^m_{\mathbf{C}^*}(E)$, then $\mathcal{F}(a)[d] \in \mathcal{M}^m_{\mathbf{C}^*}(\check{E})$. iii) $\mathcal{F} \circ \mathcal{F}[2d](d)$ is the identity functor.

5. Composing Correspondences

We will need only the following special case of composing correspondences.

Proposition 5.1. *Let $X \xleftarrow{f_1} E \xrightarrow{g_1} Y$, $Y \xleftarrow{f_2} F \xrightarrow{g_2} Z$ be two correspondences with f_i smooth and g_i proper. Put $R = E \times_Y F$, and suppose the natural map $R \to Y \times Z$ is a closed embedding, and that $E \times Z$ intersects $X \times F$ transversely in $X \times Y \times Z$. Write $X \xleftarrow{f} R \xrightarrow{g} Z$ for the induced correspondence (f is smooth, g is proper). Then*

$$I_{f_2,g_2} \circ I_{f_1,g_1} = I_{f,g} : D^{\mathbf{C}^*}_{\mathrm{qcoh}}(T^*X) \to D^{\mathbf{C}^*}_{\mathrm{qcoh}}(T^*Z).$$

6. Almost Hopf algebra $U_q\mathfrak{n}$.

In the remaining part of this paper we will apply the previous constructions to the moduli space of representations of a fixed quiver. We begin by recalling Lusztig's geometric construction of $U_q\mathfrak{n}$, the $-$part of the quantized enveloping algebra attached by Drinfeld and Jimbo to the generalised Cartan matrix (a_{ij}).

In [L1] unmixed complexes are used (i.e. $\mathcal{D}$-modules as opposed to mixed Hodge modules) and an algebra over $\mathbf{Z}[v, v^{-1}]$ is produced, where v acts by shifting the complex. When using mixed complexes, our bialgebras live naturally inside complexes of weight zero, and so we produce algebras over $\mathbf{Z}[v^2, v^{-2}]$, where v^2 acts by shift of 2 and Tate twist. This results in some small difference in formulae: If $\mathcal{L}_Z$ is some Hodge module which is the perverse extension of the constant sheaf on a smooth subvariety Z, it naturally has weight $\dim Z$.

6.1 Suppose given two finite sets I and Ω with I non-empty, a map $\Omega \to I$ denoted $h \mapsto h'$ and a map $\Omega \to I$ denoted $h \mapsto h''$, such that $h' \neq h''$ for all $h \in \Omega$.

Let $\mathbf{V}$ be a finite dimensional I-graded vector space over $\mathbf{C}$. Let $\mathbf{E_V} = \oplus_{h\in\Omega}\mathrm{Hom}(V_{h'}, V_{h''})$. The algebraic group $G_\mathbf{V} = \prod_i \mathrm{Aut}(\mathbf{V}_i)$ acts on $\mathbf{E_V}$ by $(g, x) \to gx = x'$ where $x'_h = g_{h''}x_hg_{h'}^{-1}$ for all $h \in \Omega$.

Let $\mathcal{P}_\mathbf{V}$ be the set of isomorphism classes of simple objects in $\mathcal{M}^m_{G_\mathbf{V}}(\mathbf{E_V})$, defined as in [L1, 2.2] (i.e. those L for which $L[d](e)$ appears as a direct summand of $L_{\mathbf{i},\mathbf{a}}$ for some $(\mathbf{i}, \mathbf{a}) \in S_\nu$ and some $d, e \in \mathbf{Z}$; notation as in [L1,2.2]). Let $\mathcal{Q}_\mathbf{V}$ be the set of isomorphism classes of complexes in $D\mathcal{M}^m_{G_\mathbf{V}}(\mathbf{E_V})$ of weight 0 which are isomorphic to a direct sum of finitely many complexes of the form $L[d](e)$, for $L \in \mathcal{P}_\mathbf{V}$. Note that $wt(L) - d + 2e = 0$. See (7.1) for another description of $\mathcal{Q}_\mathbf{V}$.

Let $\underline{\mathcal{K}}_\mathbf{V}$ to be the abelian group with one generator (L) for each element of $\mathcal{Q}_\mathbf{V}$ and with relations $(L) + (L') = (L'')$ whenever L'' is isomorphic to $L \oplus L'$. We regard $\underline{\mathcal{K}}_{\mathbf{V}}$ as a module over $\mathbf{Z}[v^2, v^{-2}]$ (where v is an indeterminate) by $v^n(L) = (L[2n](n))$. Clearly $\underline{\mathcal{K}}_{\mathbf{V}}$ is a free $\mathbf{Z}[v^2, v^{-2}]$-module with basis $(L[wtL])$ where L runs over $\mathcal{P}_{\mathbf{V}}$. Also write $\mathcal{K}_{\mathbf{V}} = \mathbf{Z}[v, v^{-1}] \otimes_{\mathbf{Z}[v^2,v^{-2}]} \underline{\mathcal{K}}_{\mathbf{V}}$.

6.2 Let $\mathbf{V}''$ be an I-graded subspace of $\mathbf{V}$, and $\mathbf{V}' = \mathbf{V}/\mathbf{V}''$ the I-graded quotient.

Let P be the stabilizer of $\mathbf{V}''$ in $G_{\mathbf{V}}$. This is a parabolic subgroup of $G_{\mathbf{V}}$. Its unipotent radical is denoted U. We have canonically, $P/U = G_{\mathbf{V}'} \times G_{\mathbf{V}''}$.

Let F be the closed subvariety of $\mathbf{E}_{\mathbf{V}}$ consisting of all $x \in \mathbf{E}_{\mathbf{V}}$ such that $x_h(\mathbf{V}''_{h'}) \subset \mathbf{V}''_{h''}$ for all $h \in \Omega$. We denote $i : F \to \mathbf{E}_{\mathbf{V}}$ the inclusion. We can regard $\mathbf{E}_{\mathbf{V}}$ as a vector bundle over F (after choosing a subspace of $\mathbf{V}$ isomorphic to $\mathbf{V}'$ and complementary to $\mathbf{V}''$) and then i is the zero section embedding. Note that P acts on F (restriction of the $G_{\mathbf{V}}$-action on $\mathbf{E}_{\mathbf{V}}$).

If $x \in F$, then x induces elements $x' \in \mathbf{E}_{\mathbf{V}'}$ and $x'' \in \mathbf{E}_{\mathbf{V}''}$; the map $x \mapsto (x', x'')$ is a vector bundle $\phi : F \to \mathbf{E}_{\mathbf{V}'} \times \mathbf{E}_{\mathbf{V}''}$. Now P acts on $\mathbf{E}_{\mathbf{V}'} \times \mathbf{E}_{\mathbf{V}''}$ through its quotient $P/U = G_{\mathbf{V}'} \times G_{\mathbf{V}''}$. The map ϕ is compatible with the P-actions.

We set $G_{\mathbf{V}} = G, P/U = \bar{G}$, $\mathbf{E}_{\mathbf{V}} = E$, $\mathbf{E}_{\mathbf{V}'} \times \mathbf{E}_{\mathbf{V}''} = \bar{E}$. We have a diagram

$$\bar{E} \xleftarrow{\phi} F \xrightarrow{i} E.$$

and

$$G \times_P \bar{E} \xleftarrow{p} G \times_P F \xrightarrow{q} E$$

where p is smooth with affine space fibres, and q is proper. Also write $\bar{E} \xrightarrow{r} G \times_P \bar{E}$ for the inclusion.

6.3 Let $A' \in \mathcal{Q}_{\mathbf{V}'}$, $A'' \in \mathcal{Q}_{\mathbf{V}''}$. Define $A' * A'' \in \mathcal{Q}_{\mathbf{V}}$ by $A' * A'' = q_! p^* A$ where $A \in \mathcal{M}_G^m(G \times_P \bar{E})$ is the unique complex with $r^* A = A' \boxtimes A''$. (Observe that $q_!, p^*$ and r^* all preserve weights, and so $A' * A''$ is of weight 0; [L1,3.2] shows it is really in $\mathcal{Q}_{\mathbf{V}}$). This induces a linear map $\underline{\mathcal{K}}_{\mathbf{V}'} \otimes \underline{\mathcal{K}}_{\mathbf{V}''} \to \underline{\mathcal{K}}_{\mathbf{V}}$; we also define a modified such map $\circ : \mathcal{K}_{\mathbf{V}'} \otimes \mathcal{K}_{\mathbf{V}''} \to \mathcal{K}_{\mathbf{V}}$ by $A' \circ A'' = v^m A' * A''$, where

$$-m = \sum_{h \in \Omega} \dim \mathbf{V}'_{h'} \dim \mathbf{V}''_{h''} + \sum_{i \in I} \dim \mathbf{V}'_i \dim \mathbf{V}''_i.$$

6.4 Let $B \in \mathcal{Q}_{\mathbf{V}}$. Define $\mathrm{res}_{\mathbf{V}',\mathbf{V}''} B \in \mathcal{Q}_{\mathbf{V}'} \boxtimes \mathcal{Q}_{\mathbf{V}''}$ by $\mathrm{res}_{\mathbf{V}',\mathbf{V}''} B = \phi_!(i^* B)$. As B is monodromic (even equivariant) $\mathrm{res}_{\mathbf{V}',\mathbf{V}''}$ is of weight 0 (sees §4), and by [L1,4.2] is actually in $\mathcal{Q}_{\mathbf{V}'} \boxtimes \mathcal{Q}_{\mathbf{V}''}$. This induces a linear map $\mathrm{res}_{\mathbf{V}',\mathbf{V}''} : \underline{\mathcal{K}}_{\mathbf{V}} \to \underline{\mathcal{K}}_{\mathbf{V}'} \otimes \underline{\mathcal{K}}_{\mathbf{V}''}$; we also define a modified linear map $\widetilde{\mathrm{res}}_{\mathbf{V}',\mathbf{V}''} : \mathcal{K}_{\mathbf{V}} \to \mathcal{K}_{\mathbf{V}'} \otimes \mathcal{K}_{\mathbf{V}''}$ by $\widetilde{\mathrm{res}}_{\mathbf{V}',\mathbf{V}''} B = v^s \mathrm{res}_{\mathbf{V}',\mathbf{V}''} B$, where $s = \sum_{h \in \Omega} \dim \mathbf{V}'_{h'} \dim \mathbf{V}''_{h''} - \sum_{i \in I} \dim \mathbf{V}'_i \dim \mathbf{V}''_i$.

6.5 We now recall the main theorem of [L1] (see also [L2,GL]).

Let $U_q\mathfrak{n}$ be the $\mathbf{Q}(v)$-algebra with generators F_i, $i \in I$, and relations

$$\sum_{0 \le r \le n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} F_i^r F_j F_i^{n-r} = 0,$$

where $i \neq j$, $n = 1 - a_{ij}$, and the generalised Cartan matrix (a_{ij}) is defined by $a_{ii} = 2$, $-a_{ij}$ is the number of $h \in \Omega$ such that $\{h', h''\} = \{i, j\}$ as unordered set.

Theorem. *There is an algebra isomorphism $U_q\mathfrak{n} \to \oplus_{\mathbf{V}} \mathbf{Q}(v) \otimes_{\mathbf{Z}[v,v^{-1}]} \mathcal{K}_{\mathbf{V}}$, where multiplication on the direct sum is given by $(A', A'') \to A' \circ A''$, and $\mathbf{V}$ runs over isomorphism classes of I-graded vector spaces, i.e over $\mathbf{N}^I$. Further, $\widetilde{res}_{\tau,\omega} u = \Delta_{\tau,\omega} u$, where $\Delta(u) = \sum (1 \otimes K_{-\tau}) \Delta_{\tau,\omega} u$ for $u \in U_q\mathfrak{n}$ and Δ is the usual comultiplication.*

We remark that it is essential (though trivial) to check that [L1,5-7] remain true in our mixed category. We use the Fourier transform of (4.5) rather than Deligne-Fourier transform. Also, one may ask if $L \in \mathcal{P}_{\mathbf{V}}$ has $wt(L) = \dim \operatorname{supp} L$; this is true for

7. Almost Hopf algebra $U_q\widehat{\mathfrak{n}}$

7.1 Let $\Lambda_{\mathbf{V}} \subseteq T^*\mathbf{E_V}$ be the variety defined in [L1,12.1]. That is, identify $T^*\mathbf{E_V}$ with $\mathrm{Hom}(\mathbf{E_V}, \mathbf{E_V})$ and say $x \in T^*\mathbf{E_V}$ is nilpotent if the corresponding endomorphism of $\mathbf{E_V}$ is. Recall $G_{\mathbf{V}}$ acts on $\mathbf{E_V}$ and so we have the moment map $T^*\mathbf{E_V} \xrightarrow{\mu} \mathfrak{g}_{\mathbf{V}}^*$ to the dual of the Lie algebra of $G_{\mathbf{V}}$. Then

$$\Lambda_{\mathbf{V}} = \{x \in T^*\mathbf{E_V} \mid \mu(x) = 0 \text{ and } x \text{ is nilpotent }\}.$$

For example, if (I, Ω) is a Dynkin diagram of finite type, then $\Lambda_{\mathbf{V}} = \mu^{-1}(0)$ and $\Lambda_{\mathbf{V}}$ is just the union of conormal bundles to the $G_{\mathbf{V}}$ orbits in $\mathbf{E_V}$ [L1,14.2]. In general, $\Lambda_{\mathbf{V}}$ is Lagrangian [L1,12.9].

Note that the elements of $\mathcal{Q}_{\mathbf{V}}$ (defined in (6.1)) are objects in $D\mathcal{M}^m_{G_{\mathbf{V}}}(\mathbf{E_V}, \Lambda_{\mathbf{V}})$ (i.e. if $A \in \mathcal{Q}_{\mathbf{V}}$, $\widetilde{\mathrm{gr}}A$ has support in $\Lambda_{\mathbf{V}}$ [L1,13.6]). One may in fact show that $\mathcal{Q}_V$ is precisely the isomorphism classes of weight 0 complexes in $D\mathcal{M}^m_{G_{\mathbf{V}}}(\mathbf{E_V}, \Lambda_{\mathbf{V}})$.

7.2 Write $\widehat{\underline{\mathcal{K}}}_{\mathbf{V}} = K^{G_{\mathbf{V}} \times \mathbf{C}^*}(\Lambda_{\mathbf{V}})$; this is a $\mathbf{Z}[v^2, v^{-2}]$-module, on identifying v^2 with the canonical generator of $K_{\mathbf{C}^*}$ (so $K_{\mathbf{C}^*} = \mathbf{Z}[v^2, v^{-2}]$). Also set $\widehat{\mathcal{K}}_{\mathbf{V}} = \mathbf{Z}[v, v^{-1}] \otimes_{\mathbf{Z}[v^2, v^{-2}]} \widehat{\underline{\mathcal{K}}}_{\mathbf{V}}$

7.3 We retain the notation of (6.2). Define a multiplication $* : \widehat{\underline{\mathcal{K}}}_{\mathbf{V}'} \otimes \widehat{\underline{\mathcal{K}}}_{\mathbf{V}''} \to \widehat{\underline{\mathcal{K}}}_{\mathbf{V}}$ by $a' * a'' = I_{p,q}a$, where $a \in M^{G \times \mathbf{C}^*}_{\mathrm{coh}}(T^*(G \times_P \bar{E}))$ is induced from $a' \boxtimes a''$ by the procedure of (3.1).

Also, define a modified multiplication $\circ : \widehat{\mathcal{K}}_{\mathbf{V}'} \otimes \widehat{\mathcal{K}}_{\mathbf{V}''} \to \widehat{\mathcal{K}}_{\mathbf{V}}$ by $a' \circ a'' = v^m a' * a''$, where m is as in (6.3). One may easily verify that $*$ (and hence $\circ$) is associative. This follows from the diagrams in [L1,3.4] and (5.1), once we verify the assumptions of that proposition. This is trivial.

7.4 We have thus defined an algebra structure on $\oplus \widehat{\mathcal{K}}_{\mathbf{V}}$; and the results of §2-4 imply that $\widetilde{\mathrm{gr}} : \oplus \mathcal{K}_{\mathbf{V}} \to \oplus \widehat{\mathcal{K}}_{\mathbf{V}}$ is an algebra homomorphism. It remains to describe $\oplus \widehat{\mathcal{K}}_{\mathbf{V}}$ in terms of generators and relations.

7.5 Let $U_q\mathfrak{g}$ be the quantized enveloping algebra attached to the generalised symmetric Cartan matrix (a_{ij}). Write $U_q\widehat{\mathfrak{g}}$ for the 'affinisation' of this algebra defined in [D] (see also [B]). Actually, in [D] this is only done for $\mathfrak{g}$ of finite type, but it is trivial to observe that the small modification of this definition, below, gives for arbitrary Kac-Moody $\mathfrak{g}$, the q-analogue of the enveloping algebra of the central extension of $\mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]$ by the cocycle determined by an invariant non-degenerate bilinear form on $\mathfrak{g}$. Then $U_q\widehat{\mathfrak{g}}$ is also a Hopf algebra (see [B]). Let $U_q\widehat{\mathfrak{n}}$ be the $-$part of this algebra. This is a $\mathbf{Q}(v)$-algebra with generators $\tilde{F}_i(n)$ for $i \in I$, $n \in \mathbf{Z}$, and relations

$$(w - vz)^{-a_{ij}} \tilde{F}_i(z)\tilde{F}_j(w) = (wv - z)^{-a_{ij}} \tilde{F}_j(w)\tilde{F}_i(z) \tag{7.6.1i}$$

where $\tilde{F}_i(z) = \sum \tilde{F}_i(n)z^n$, $i, j \in I, i \neq j$, and

$$(wv^2 - z)\tilde{F}_i(z)\tilde{F}_i(w) = (w - v^2z)\tilde{F}_i(w)\tilde{F}_i(z) \tag{7.6.1ii}$$

and, for $i, j \in I$, $i \neq j$, and $k_1, \dots k_n, l \in \mathbf{Z}$

$$(7.6.2) \qquad \sum_{\sigma \in S_n} \sum_{0 \le r \le n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} \tilde{F}_i(k_{\sigma 1}) \cdots \tilde{F}_i(k_{\sigma r}) \tilde{F}_j(l) \tilde{F}_i(k_{\sigma(r+1)}) \cdots \tilde{F}_i(k_{\sigma n}) = 0$$

where $n = 1 - a_{ij}$, $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$ has its usual meaning [L2,1.3.1] and S_n denotes the symmetric group on n letters (so there are $(n+1)!$ terms in the above relation).

It will be convenient for us to slightly modify the generators $\tilde{F}_i(n)$. Suppose that Ω has no cycles of odd length, and that all $-a_{ij}$ arrows from i to j in Ω are pointing in the same direction. Fix, once and for all, a function γ_i such that $\gamma_{h'} = \gamma_{h''} + 1$ if $h \in \Omega$. Define $F_i(n)$ by the relation $\sum F_i(n) z^n = \tilde{F}_i(v^{\gamma_i} z)$. These new elements are clearly generators of $U_q \widehat{\mathfrak{n}}$.

Write $\mathbf{U}^0$ for the $\mathbf{Q}(v)$-group algebra of the free abelian group generated by K_i, $i \in I$. Then $\mathbf{U}^0 \otimes U_q \widehat{\mathfrak{n}}$ is an algebra, where we define $K_i F_j(n) K_i^{-1} = v^{-a_{ij}} F_j(n)$.

For $\nu \in \mathbf{N}^I$ define

$$U_q \widehat{\mathfrak{n}}_\nu = \{ u \in U_q \widehat{\mathfrak{n}} \mid K_i u K_i^{-1} = v^{-\sum \nu_j a_{ij}} u \},$$

so $U_q \widehat{\mathfrak{n}} = \oplus_{\nu \in \mathbf{N}^I} U_q \widehat{\mathfrak{n}}_\nu$.

7.7 Let $\mathbf{V}(i)$ be an I-graded vector space with $\dim \mathbf{V}(i)_j = 0$ if $i \neq j$, $\dim \mathbf{V}(i)_i = 1$. Then $\underline{\widehat{\mathcal{K}}}_{\mathbf{V}(i)}$ is canonically a Laurent polynomial ring, $\mathbf{Z}[v^2, v^{-2}, r, r^{-1}]$, where as always v^2 represents the canonical generator of $K^{\mathbf{C}^*}$, and r denotes the canonical generator of $K^{G_{\mathbf{V}(i)}}$. Denote r^n by $F_i(n)$.

Theorem 7.8. *There is an algebra isomorphism* $\phi : U_q \widehat{\mathfrak{n}} \to \oplus_{\mathbf{V}} \mathbf{Q}(v) \otimes_{\mathbf{Z}[v, v^{-1}]} \widehat{\mathcal{K}}_{\mathbf{V}}$ *where the sum is over isomorphism classes of* I*-graded vector spaces (i.e. over* $\mathbf{N}^I$*), and multiplication on the direct sum is given by* $(a', a'') \mapsto a' \circ a''$*. This isomorphism identifies* $U_q \widehat{\mathfrak{n}}_\nu$ *with* $\mathbf{Q}(v) \otimes_{\mathbf{Z}[v,v^{-1}]} \widehat{\mathcal{K}}_{\mathbf{V}}$ *where* $\dim \mathbf{V} = \nu$*, and is defined by sending* $F_i(n)$ *to the element* $F_i(n) \in \widehat{\mathcal{K}}_{\mathbf{V}(i)}$ *described in (7.7).*

Proof. We will show surjectivity of ϕ in (7.10), and injectivity in §9. To show ϕ is a homomorphism, we must check that the relations (7.6.1),(7.6.2) hold in $\oplus \widehat{\mathcal{K}}_{\mathbf{V}}$. We describe the "Drinfeld-Serre relation" (7.6.2); the "exchange condition" (7.6.1) is easier and left to the reader (see also [G1]).

Observe (7.6.2) is an identity that must be verified in $\widehat{\mathcal{K}}_{\mathbf{W}(i,j)}$, where $\mathbf{W}(i,j)$ is an I-graded vector space with $\dim \mathbf{W}(i,j)_k = 0$, $k \notin \{i, j\}$, $\dim \mathbf{W}(i,j)_j = 1$, $\dim \mathbf{W}(i,j)_i = n = 1 - a_{ij}$. Set $G = G_{\mathbf{W}(i,j)} \simeq GL_n \times GL_1$, and $T = (GL_1)^n \times GL_1$, a maximal torus of G. Let $R(k_1, \dots, k_n, l) = \sum_{\sigma \in S_n} Ind_T^G (L_{k_{\sigma 1}} \boxtimes \cdots \boxtimes L_{k_{\sigma n}} \boxtimes L_l)$, where $L_{k_1} \boxtimes \cdots \boxtimes L_{k_n} \boxtimes L_l$ is the one dimensional T-module where $(z_1, \dots, z_n, w)$ acts by multiplication by $z_1^{k_1} \cdots z_n^{k_n} w^l$, and Ind is induction in the sense of K-theory. Then $R(k_1, \dots, k_n, l) \in K^{G \times \mathbf{C}^*}$ and the character of $R(k_1, \dots, k_n, l)$ is $\sum_{\sigma \in S_n} L_{k_{\sigma 1}} \boxtimes \cdots \boxtimes L_{k_{\sigma n}} \boxtimes L_l$ (by the projection formula in K-theory).

More generally, we have

$$\sum_{\sigma \in S_n} F_i(k_{\sigma 1}) \circ \cdots \circ F_i(k_{\sigma r}) \circ F_j(l) \circ F_i(k_{\sigma(r+1)}) \circ \cdots \circ F_i(k_{\sigma n})$$
$$= F_i(0)^r \circ F_j(0) \circ F_i(0)^{n-r} \otimes R(k_1, \dots, k_n, l)$$

from the definition of the multiplication $\circ$, and by the projection formula in K-theory. So to prove (7.6.2) it suffices to prove it for $(k_1, \ldots, k_n, l) = (0, \ldots, 0)$.

But it is clear that if F_i denotes the generator of $\mathcal{K}_{\mathbf{V}(i)}$, $\widetilde{\mathrm{gr}} F_i = F_i(0)$. Hence as $\widetilde{\mathrm{gr}}$ commutes with $\circ$, and we know that the usual q-Serre relation

$$\sum_r (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} F_i^r F_j F_i^{n-r} = 0$$

holds in $\oplus_{\mathbf{V}} \mathcal{K}_{\mathbf{V}}$, we have the result.

One may also prove that ϕ is a homomorphism directly in K-theory. Then, remarkably, the restriction to Λ^{reg} is already sufficent to determine the coefficents of any linear relations between the monomials in the generators $F_i(k)$. We thus have an overdetermined system of relations, and it is a fact that $0 = \Lambda^{sing}$ does not contribute. For example, if $a_{ij} = -1$, there are 16 terms which sum to zero (see (9.5)). On the other hand, it is also reasonable to think of *all* the action as taking place at 0, see §8 below.

One may also define a $\mathbf{Z}[v, v^{-1}]$ form of $U_q\widehat{\mathfrak{n}}$ (analagous to [L2]); it is the subaglebra generated by elements $F_i^{(k)}(n) = F_i(n)^k/[k]!$. Then we have a homomorphism of almost Hopf algebras from this $\mathbf{Z}[v, v^{-1}]$-form to $\oplus_{\mathbf{V}} \widehat{\mathcal{K}}_{\mathbf{V}}$, and the proof below shows that this is an isomorphism too.

7.9 Following [L1,12.2] for $k \in I$ and $p \in \mathbf{N}$, put

$$\Lambda_{\mathbf{V},k,p} = \{(f_h) \in \Lambda_{\mathbf{V}} \mid \operatorname{codim}_{\mathbf{V}_k}(\sum_{h \in H : h'' = k} im(f_h : \mathbf{V}_{h'} \to \mathbf{V}_k)) = p\}$$

Then $\overline{\Lambda_{\mathbf{V},k,p}}$ is a union of components of $\Lambda_{\mathbf{V}}$ [L1,12.3], and every component of $\Lambda_{\mathbf{V}}$ lies in some $\overline{\Lambda_{\mathbf{V},k,p}}$ with $p > 0$ [L1,12.6]. Also, $\overline{\Lambda_{\mathbf{V},k,p}} = \cup_{p' \geq p} \Lambda_{\mathbf{V},k,p}$.

7.10 We now prove ϕ is surjective. We first show that $\oplus_{\mathbf{V}} \widehat{\mathcal{K}}_{\mathbf{V}}$ is generated by $\widehat{\mathcal{K}}_{n\mathbf{V}(i)}$, $i \in I$, $n \in \mathbf{N}$ (where $\dim(n\mathbf{V})_i = n \dim \mathbf{V}_i$).

Let $\mathbf{V}$ be an I-graded vector space, and suppose we know the result for all such $\mathbf{V}'$, $\dim \mathbf{V}' < \dim \mathbf{V}$. Let a be a $G \times \mathbf{C}^*$-equivariant coherent sheaf on $\Lambda_{\mathbf{V}}$. We can assume the support of a, $\operatorname{supp}(a)$, is contained in some irreducible component of $\Lambda_{\mathbf{V}}$. (For if $\operatorname{supp}(a) \subseteq X \cup Y$, X, Y two components, let $X' = X \setminus (X \cap Y)$, $j : X' \hookrightarrow X \cup Y$ the inclusion of this open subvariety. Then there exists an a' on $\overline{X'}$ such that $j^*(a') = j^* a$ [Th], and $a' - a$ has support in Y).

So $\operatorname{supp}(a) \subseteq \overline{\Lambda_{\mathbf{V},k,p}}$ for some $k \in I$, $p > 0$. We can further assume inductively that multiplication is surjective to those sheaves a with $\operatorname{supp}(a) \subseteq \overline{\Lambda_{\mathbf{V},k,p'}}$ for $p' > p$ (as $\Lambda_{\mathbf{V},k,p'} = 0$ for $p' >> 0$).

Fix a subspace $W \subseteq \mathbf{V}_k$ of codimension p. Let $\mathbf{V}''$ be the I-graded vector space with $\mathbf{V}''_i = \mathbf{V}_i$, $i \neq k$, $\mathbf{V}''_k = W$, and put $\mathbf{V}' = \mathbf{V}/\mathbf{V}''$. Let $E, \bar{E}, F, P, G, \bar{G}$ be as in 6.2; note $F = \bar{E}$, $\bar{G} = GL_{\mathbf{V}''} \times GL_{(\mathbf{V}_k/W)}$. Let $j : \Lambda_{\mathbf{V},k,p} \hookrightarrow \Lambda_{\mathbf{V}}$ denote the inclusion. Then if

$$T^*(G \times_P \bar{E}) \xleftarrow{r} T^*_F(G \times_P \bar{E} \times E) \xrightarrow{s} T^* E$$

denotes the correspondence associated to this data, a short calculation shows that when restricted to $s^{-1}(\Lambda_{\mathbf{V},k,p})$, s is an inclusion and r is an affine space fibration to $\Lambda_{\mathbf{V}'',k,0}$.

It follows that there exists a $\bar{G} \times \mathbf{C}^*$-equivariant coherent sheaf a'' on $\Lambda_{\mathbf{V}''}$ with support $\Lambda_{\mathbf{V}'',k,0}$ such that if a' denotes the induced $G \times \mathbf{C}^*$-equivariant coherent sheaf on $G \times_P \Lambda_{\mathbf{V}''}$, $j^*(\omega \otimes r^* a') = j^* a$. Here $\omega = \omega_{G \times_P \bar{E}/E}$ is a line bundle (and so tensoring with ω induces an isomorphism in K-theory). Thus $\operatorname{supp}(a - a_1 * a_2) \subseteq \overline{\Lambda_{\mathbf{V},k,p}} - \Lambda_{\mathbf{V},k,p}$, where $a_1 \boxtimes a_2 = a''$, $a_1 \in K^{GL_{\mathbf{V}''} \times \mathbf{C}^*}(\Lambda_{\mathbf{V}''})$, $a_2 \in K^{GL_{\mathbf{V}_k/W} \times \mathbf{C}^*}$ and $*$ denotes the multiplication of 7.3.

By induction on p, we are done.

7.11 It remains to show $\oplus_n \mathbf{Q}(v) \otimes_{\mathbf{Z}[v,v^{-1}]} \widehat{\mathcal{K}}_{n\mathbf{V}(i)}$ is generated by $F_i(a)$, $a \in \mathbf{Z}$. But $\sum_{\sigma \in S_n} F(a_{\sigma 1}) * \cdots * F(a_{\sigma n}) = C m_{(a_n, \ldots, a_1)}$ where $C = \prod_{i \leq n} \frac{v^{2n}-1}{v^2-1}$ and $m_\lambda = \sum_{w \in S_n} e^{w\lambda}$ denotes the monomial symmetric sum. As these clearly span K^{GL_n}, we are done.

In fact, $F(a_{\sigma 1}) * \cdots * F(a_{\sigma n})$ is essentially a Hall-Littlewood polynomial, and a more careful analysis shows $\oplus \widehat{\mathcal{K}}_{n\mathbf{V}(i)}$ is generated by $F_i^{(n)}(a) = F_i(a)^n/[n]!$, the representation $det^{\otimes a}$ of GL_n (see [G1]).

8. Quasi-particle construction

In this section we eliminate Λ, defining multiplication on the equivariant cohomology of a point. The same trick, when applied to the affine Hecke algebra, realises it as an algebra of $q-$BGG operators.

8.1 Define $\underline{\mathbf{U}}_{QP} = \oplus_{\mathbf{V} \in \mathbf{N}^I} K^{G_{\mathbf{V}} \times \mathbf{C}^*}$, and $\mathbf{U}_{QP} = \underline{\mathbf{U}}_{QP} \otimes_{\mathbf{Z}[v^2,v^{-2}]} \mathbf{Z}[v, v^{-1}]$. Given $\mathbf{V} \in \mathbf{N}^I$, and $\mathbf{V}'' \leq \mathbf{V}$ an I-graded subspace as in 6.2, put $\mathbf{V}' = \mathbf{V}/\mathbf{V}''$ and, retaining the notation of (6.2), put

$$P = \{x \in G_{\mathbf{V}} \mid x_i \mathbf{V}_i'' \subseteq \mathbf{V}_i'', i \in I\}, \qquad \mathfrak{n} = \{x \in \mathfrak{g}_{\mathbf{V}} \mid x_i \mathbf{V}_i'' = 0, x_i \mathbf{V}_i \subseteq \mathbf{V}_i'', i \in I\}$$

$$R = \{x \in E \mid x_h \mathbf{V}_{h'}'' = 0, x_h \mathbf{V}_{h'} \subseteq \mathbf{V}_{h''}'', h \in \Omega\} \qquad Q = \{y \in E^* \mid y_h \mathbf{V}_{h'}'' = 0, y_h \mathbf{V}_{h'} \subseteq \mathbf{V}_{h''}'', h \in \bar{\Omega}\}.$$

Observe that Q, R, $\mathfrak{n}$ are all representations of P in a natural way.

Let $\mathcal{A}_1$ be a representation of $G_{\mathbf{V}'} \times \mathbf{C}^*$, $\mathcal{A}_2$ be a representation of $G_{\mathbf{V}''} \times \mathbf{C}^*$, so $\mathcal{A}_2 \boxtimes \mathcal{A}_1$ is a representation of $P \times \mathbf{C}^*$. If A is a vector space, write $\Lambda_{-1}(A)$ for the formal linear combination $\sum (-1)^i \Lambda^i A$. If A is a P-module, $\Lambda_{-1}(A)$ is also. Write $\operatorname{Ind}_P^G : K^P \to K^G$ for the composite map $K^P = K^G(G/P) \xrightarrow{\pi_*} K^G$, where π is the map from G/P to a point, and G/P is assumed compact. This is just the usual (derived functor) of induction of P-modules.

We define $\mathcal{A}_1 * \mathcal{A}_2 \in K^{G \times \mathbf{C}^*}$ by

$$\mathcal{A}_1 * \mathcal{A}_2 = (-1)^{\dim E - \dim \bar{E}} v^{-2 \dim Q} \operatorname{Ind}_{P \times \mathbf{C}^*}^{G \times \mathbf{C}^*} (\mathcal{A}_2 \boxtimes \mathcal{A}_1 \otimes \Lambda_{-1}(v^2 \mathfrak{n} + v^{-2} R + Q^*)).$$

It is easy to see that $*$ defines an associative operation (as $\Lambda_{-1}(X + Y) = \Lambda_{-1}(X) \otimes \Lambda_{-1}(Y)$, and induction is transitive. This is obviously the same argument as that used to show the multiplication of 7.3 is associative). We also define an algebra structure $\circ$ on $\mathbf{U}_{QP}$, by $\mathcal{A}_1 \circ \mathcal{A}_2 = v^m \mathcal{A}_1 * \mathcal{A}_2$, m as in 6.3. This is still associative.

Lemma 8.2. *The map $\underline{\widehat{\mathcal{K}}}_{\mathbf{V}} \to K^{G_{\mathbf{V}} \times \mathbf{C}^*}$ given by*

$$K^{G_{\mathbf{V}} \times \mathbf{C}^*}(\Lambda_{\mathbf{V}}) \xrightarrow{i^* j_*} K^{G_{\mathbf{V}} \times \mathbf{C}^*}(\mathbf{E}_{\mathbf{V}}) \to K^{G_{\mathbf{V}} \times \mathbf{C}^*}$$

is an algebra homomorphism. Here $i : \mathbf{E_V} \hookrightarrow T^*\mathbf{E_V}$, $j : \Lambda_\mathbf{V} \hookrightarrow T^*\mathbf{E_V}$ are the natural inclusions, and the last map is induced by the homotopy invariance of K-theory (Thom isomorphism).

The lemma follows from the definition of multiplication in $\widehat{\underline{\mathcal{K}}}_\mathbf{V}$ and $K^{G_\mathbf{V}\times \mathbf{C}^*}$ and the "excess intersection" formula in K-theory; it is an easy diagram chase and omitted (see [G1]). In fact, this map is an injection. We will show this by showing that the map $U_q\widehat{\mathfrak{n}} \to \mathbf{U}_{QP}$ is an injection

8.3 To do this, define a map $\tilde\Delta_{\mathbf{V}',\mathbf{V}''} : K^{G\times \mathbf{C}^*} \to \widehat{K^{P\times\mathbf{C}^*}}$, where $\widehat{K^{P\times\mathbf{C}^*}}$ denotes the representation ring copleted at the identity of $P\times\mathbf{C}^*$. This map, which we will eventually see as the coproduct, is defined by

$$\tilde\Delta_{\mathbf{V}',\mathbf{V}''}\mathcal{A} = (-1)^{\dim E-\dim \bar E} v^{2\dim R}\mathrm{Res}^{G\times\mathbf{C}^*}_{P\times\mathbf{C}^*}(\mathcal{A})\otimes S(v^2\mathfrak{n}+R+v^{-2}Q^*)\otimes\Lambda_{-1}(\mathfrak{n}^*)$$

where $\mathcal{A}\in K^{G\times\mathbf{C}^*}$, $\mathrm{Res}^G_P : K^G\to K^P$ is the forgetful functor, and $SX = \oplus_n S^nX$ is the symmetric algebra. Identifying $K^{P\times\mathbf{C}^*} = K^{G_{\mathbf{V}'}\times\mathbf{C}^*}\otimes_{\mathbf{Z}[v^2,v^{-2}]} K^{G_{\mathbf{V}''}\times\mathbf{C}^*}$, we have defined a map

$$\tilde\Delta : \underline{\mathbf{U}}_{QP}\to \underline{\mathbf{U}}_{QP}\hat\otimes\underline{\mathbf{U}}_{QP}, \qquad \tilde\Delta(\mathcal{A}) = \sum \tilde\Delta_{\mathbf{V}',\mathbf{V}''}(\mathcal{A})$$

where the sum is over pairs $\tau,\nu\in\mathbf{N}^I$ such that $\tau+\nu=\dim\mathbf{V}$, and $\mathbf{V}''$ is any I-graded subspace of $\mathbf{V}$ of dimension τ, $\mathbf{V}'=\mathbf{V}/\mathbf{V}''$. Define Δ, $\Delta_{\mathbf{V}',\mathbf{V}''}$ by $\Delta_{\mathbf{V}',\mathbf{V}''}(\mathcal{A}) = v^t\tilde\Delta_{\mathbf{V}',\mathbf{V}''}(\mathcal{A})$, where $t = m-2\dim E+2\dim\bar E$, m as in (6.3).

8.4 Define a $\mathbf{Q}((v))$-valued symmetric bilinear form, $(,)$ on $\underline{\mathbf{U}}_{QP}$ by requiring that $(K^{G_{\mathbf{V}_1}\times\mathbf{C}^*}, K^{G_{\mathbf{V}_2}\times\mathbf{C}^*})=0$ unless $\mathbf{V}_1=\mathbf{V}_2$, and

$$(\mathcal{A}_1,\mathcal{A}_2) = (\mathcal{A}_1\otimes\mathcal{A}_2\otimes S(v^2\mathfrak{g}+v^{-2}E+E))^G$$

if $\mathcal{A}_1,\mathcal{A}_2\in K^{G\times\mathbf{C}^*}$. Here X^G denotes the G-invariants of X. For example,

$$(F_i(n),F_i(m)) = \delta_{n,-m}(1-v^2)^{-1}.$$

Lemma 8.5. $(\mathcal{A}_1*\mathcal{A}_2,\mathcal{A}_3) = v^{-2\dim E+2\dim\bar E}(\mathcal{A}_1\otimes\mathcal{A}_2,\tilde\Delta_{\mathbf{V}',\mathbf{V}''}(\mathcal{A}_3))$.

Proof. As $\mathrm{Ind}^G_P(\mathcal{A})\otimes\mathcal{B} = \mathrm{Ind}^G_P(\mathcal{A}\otimes\mathrm{Res}^G_P\mathcal{B})$ if $\mathcal{B}$ is a representation of G, and as

$$(\mathrm{Ind}^G_P(\mathcal{A}))^G = (\mathcal{A}\otimes\Lambda_{-1}(\mathfrak{n}^*))^P$$

we have

$$\begin{aligned}&(\mathcal{A}_1*\mathcal{A}_2,\mathcal{A}_3) =\\ &= (-1)^{\dim E-\dim\bar E}v^{-2\dim Q}((\mathcal{A}_2\boxtimes\mathcal{A}_1)\otimes\mathrm{Res}^G_P(\mathcal{A}_3)\otimes\Lambda_{-1}(v^2\mathfrak{n}+v^{-2}R+Q+\mathfrak{n}^*)\otimes S(v^2\mathfrak{g}+v^{-2}E+E))^P\\ &= (-1)^{\dim E-\dim\bar E}v^{-2\dim Q}((\mathcal{A}_2\boxtimes\mathcal{A}_1)\otimes\mathrm{Res}^G_P(\mathcal{A}_3)\otimes S(v^2\mathfrak{n}^*+v^{-2}Q^*+R)\otimes\Lambda_{-1}(\mathfrak{n}^*)\otimes S(v^2\bar{\mathfrak{g}}+v^{-2}\bar E+\bar E))^{\bar G}\end{aligned}$$

where $P/U=\bar G$ and $\bar{\mathfrak{g}}$ is its Lie algebra. These equalities are immediate from the definition of $*$, the fact $S(X)\otimes\Lambda_{-1}(X)=\mathbf{C}$, and the isomorphism of $\bar G$-modules $E = Q^*+\bar E+R$. Substituting the definition of $\tilde\Delta$ we get the lemma, noting that we reversed the order of $\mathcal{A}_2\boxtimes\mathcal{A}_1$ when we identify K^P with $K^{G_{\mathbf{V}'}}\otimes K^{G_{\mathbf{V}''}}$.

8.6 To see how this "coproduct" interacts with multiplication, define $U_q\widehat{\mathfrak{h}}^-$ to be the commutative algebra generated by symbols $\psi_i(n)$, $i \in I, n > 0$. Put $\psi_i(0) = 1$, and $\psi_i(z) = \sum_{n\geq 0} \psi_i(n) z^n$.

Write $\mathbf{U}^{\leq 0} = U_q\widehat{\mathfrak{n}} \otimes U_q\widehat{\mathfrak{h}}^-$. This naturally has an algebra structure, with $U_q\mathfrak{n}$, $U_q\widehat{\mathfrak{h}}^-$ as subalgebras, when we impose the additional relations

$$(w - vz)^{-a_{ij}} \psi_i(z) \tilde{F}_j(w) = (wv - z)^{-a_{ij}} \tilde{F}_j(w) \psi_i(z)$$

$$(wv^2 - z) \psi_i(z) \tilde{F}_i(w) = (w - v^2 z) \tilde{F}_i(w) \psi_i(z)$$

if $i \neq j$. Now $\mathbf{U}^{\leq 0}$ is a (topological) Hopf algebra; we define

$$\Delta \tilde{F}_i(z) = \tilde{F}_i(z) \hat{\otimes} \psi_i(z) + 1 \hat{\otimes} \tilde{F}_i(z)$$

$$\Delta \psi_i(z) = \psi_i(z) \hat{\otimes} \psi_i(z)$$

and the completed tensor product $\hat{\otimes}$ is the

$$\lim_{\rightarrow} \lim_{\leftarrow} U^a \otimes U^b / U^a_k \otimes U^b + U^a \otimes U^b_k,$$

where U^r is the vector subspace of $\mathbf{U}^{\leq 0}$ generated by $\leq r$ products of the $\psi_i(a), F_i(a)$ ($i \in I, a \in \mathbf{Z}$), and U^r_k is the intersection of U^r with the left ideal in $\mathbf{U}^{\leq 0}$ generated by $\psi_i(a), F_i(a)$, for $a \geq k$. This coproduct was first defined by Drinfeld (if $a_{ij} = -1$). See [DF] for a written account. It is a somewhat remarkable fact that Δ really is an algebra homomorphism. See (8.7), (9.5) below.

Let $\epsilon : U_q\widehat{\mathfrak{h}}^- \to \mathbf{Q}((v))$ be the algebra homomorphism defined by $\psi_i(n) \mapsto 0$ if $n \neq 0$. Denote the map $1 \otimes \epsilon : \mathbf{U}^{\leq 0} = U_q\widehat{\mathfrak{n}} \otimes U_q\widehat{\mathfrak{h}}^- \to U_q\widehat{\mathfrak{n}}$ also by ϵ. This is no longer an algebra homomorphism. Write $\phi : U_q\widehat{\mathfrak{n}} \to \mathbf{U}_{QP}$ for the composition of ϕ of (7.8) with the homomorphism of (8.2).

Proposition 8.7. *We have* $(\phi\epsilon \hat{\otimes} \phi)\Delta = \Delta\phi : U_q\widehat{\mathfrak{n}} \to \mathbf{U}_{QP} \hat{\otimes} \mathbf{U}_{QP}$.

It is clearly enough to prove the proposition for monomials. The proof consists of writing the left hand side out explicitly. It is a sum of products of the form $(wv - z/w - vz)$. This, one observes after some unpleasant calculation, is the right hand side, which is calculated using the Weyl character formula for GL_n.

A more conceptual proof, without calculation, will be given in [G1] and the sequel to this paper.

8.8 We remark that at $q = 1$ this coproduct degenerates into the standard coproduct for $\widehat{\mathfrak{n}}$, and that $U_q\mathfrak{n}$ does *not* sit inside $\mathbf{U}^{\leq 0}$ as a Hopf subalgebra.

9. Completion of proof of (7.8)

Finally, we prove the injectivity of the map $\phi : U_q\widehat{\mathfrak{n}} \to \mathbf{U}_{QP}$.

Lemma 9.1. *There exists a unique symmetric bilinear form* $(,) : U_q\widehat{\mathfrak{n}} \times U_q\widehat{\mathfrak{n}} \to \mathbf{Q}(v)$ *satisfying*

i) $(F_i(n), F_j(m)) = \delta_{ij}\delta_{n,-m}(1 - v^2)^{-1}$,

ii) $(a_1 a_2, b) = (a_1 \otimes a_2, (\epsilon \otimes 1)\Delta b)$, $(b, a_1 a_2) = ((\epsilon \otimes 1)\Delta b, a_1 \otimes a_2)$,

iii) $(U_q\widehat{\mathfrak{n}}_{\nu'}, U_q\widehat{\mathfrak{n}}_{\nu''}) = 0$ *if* $\nu' \neq \nu''$.

The proof is a standard calculation, and omitted. It follows from (8.2), (8.5) and (8.7) that $\phi : U_q\widehat{\mathfrak{n}} \to \mathbf{U}_{QP}$ is actually an isometry, and hence that any kernel of ϕ is contained in the radical of $(,)$. To show that $(,)$ has no radical, we imitate the proof of the analagous fact for $U_q\mathfrak{n}$ (Drinfeld's theorem).

Lemma 9.2. *Suppose* $u \in U_q\widehat{\mathfrak{n}}_\eta$, *and* $\sum \eta_i \geq 2$. *Then if* $\Delta u = u \otimes \psi + 1 \otimes u$ *with* $\psi \in U_q\widehat{\mathfrak{h}}^-$, *it must be that* $u = 0$.

Proof. We introduce the rest of $U_q\widehat{\mathfrak{g}}$ at trivial central charge. Recall we have $E_i(z) = \sum E_i(n)z^n$, and $\psi_i(z) = \sum_{n\leq 0}\psi_i(n)z^{-n}$ such that $\Delta E_i(z) = E_i(z)\hat{\otimes}1 + \psi_i(z)\hat{\otimes}E_i(z)$, and $[E_i(z), F_j(w)] = (v - v^{-1})^{-1}\delta_{ij}(\delta(w/z)(\psi_i(w) - \phi_i(z)))$, where $\delta(w/z) = \sum_n w^n z^{-n}$. Write $U_q\widehat{\mathfrak{h}}^+$ for the subalgebra generated by the $\psi_i(n)$, and $U_q\widehat{\mathfrak{h}}$ for the algebra generated by $\psi_i(n), \phi_i(n)$.

It follows that $[E_i(z), u] = \phi'(z)u' + u''\psi''(z)$, where $u', u'' \in U_q\widehat{\mathfrak{n}}_{\nu-\nu_i}$ and $\phi'(z) = \sum_{a\leq 0}\lambda_a\phi_i(a)z^a \in U_q\widehat{\mathfrak{h}}^+$, $\psi''(z) = \sum_{a\geq 0}\mu_a\psi_i(a)z^a \in U_q\widehat{\mathfrak{h}}^-$, for certain $\lambda_a \in \mathbf{Q}(v), \mu_a \in \mathbf{Q}(v)$. Here, $\nu_i \in \mathbf{N}^I$ is given by $(\nu_i)_j = \delta_{ij}$.

Apply Δ to this equality, and project both sides to $U_q\widehat{\mathfrak{h}}\hat{\otimes}\mathbf{U}^{\leq 0}$. As $\nu - \nu_i \neq 0$, we get

$$\phi_i(z)\hat{\otimes}\phi'(z)u' + \phi_i(z)\hat{\otimes}u''\psi'' = \Delta\phi'(z).(1\hat{\otimes}u') + (1\hat{\otimes}u'')\Delta\psi''(z)$$

whence $u'' = 0$. But then $[E_i(z), u] = \phi'(z)u'$ implies

$$\phi'(z)u'\hat{\otimes}\psi + \phi_i(z)\hat{\otimes}\phi'(z)u = \Delta\phi'(z)\Delta u' = \Delta\phi'(z)(u'\hat{\otimes}\psi' + 1\hat{\otimes}u' + \text{other terms})$$

which is not possible unless $u' = 0$ (look at the homogeneity of the first term on each side of the equation).

It follows that $[E_i(z), u] = 0$ for all i. But this is not possible unless $u = 0$, as there exist irreducible "Verma" modules for $U_q\widehat{\mathfrak{g}}$.

9.3 Now suppose that $y \in U_q\widehat{\mathfrak{n}}_\nu$ is in the radical of $(,)$. We can assume $(,)$ is non-degenerate on weight spaces $U_q\widehat{\mathfrak{n}}_{\nu'}$, $\sum \nu_i' < \sum \nu_i$. As the form is clearly non-degenerate when $\sum \nu_i < 2$, we assume $\sum \nu_i \geq 2$. Write $(\epsilon \otimes 1)\Delta y = u \otimes \psi + 1 \otimes u + \sum_{\alpha+\beta=\nu} y_{\alpha\beta}$, where $y_{\alpha\beta} \in U_q\widehat{\mathfrak{n}}_\alpha\hat{\otimes}U_q\widehat{\mathfrak{n}}_\beta$. Then, by the inductive assumption of non-degeneracy, if $y_{\alpha\beta} \neq 0$, there exists $u_{\alpha\beta} \in U_q\widehat{\mathfrak{n}}_\alpha\hat{\otimes}U_q\widehat{\mathfrak{n}}_\beta$ with $(u_{\alpha\beta}, y_{\alpha\beta}) \neq 0$. But then $(m(u_{\alpha\beta}), y) = (u_{\alpha\beta}, \Delta y) \neq 0$, where $m : U_q\widehat{\mathfrak{n}}\otimes U_q\widehat{\mathfrak{n}} \to U_q\widehat{\mathfrak{n}}$ is the multiplication map. It follows that $y_{\alpha\beta} = 0$, i.e. that $\Delta u = u \otimes \psi + 1 \otimes u$. But then lemma 9.2 implies that $y = 0$.

Hence $(,)$ is non-degenerate, and theorem 7.8 is finally proved.

9.4 This allows us to define $U_q\widehat{\mathfrak{n}}$ as the quotient of the free associative algebra by the kernel of the radical of the form defined as in 9.1; just as with $U_q\mathfrak{n}$. We then get the Drinfeld relations (7.6) for "free". In our case, given how we must complete the tensor product to get a coproduct, this seems like a less pleasant way to proceed then the explicit relations.

9.5 It is worth writing explicitly the identity

$$\tfrac{z_2-qz_1}{qz_2-z_1}\tfrac{qw-z_1}{w-z_1} + \tfrac{qw-z_1}{w-z_1} - (q+1)\tfrac{z_2-qz_1}{qz_2-z_1}\tfrac{qw-z_1}{w-z_1}\tfrac{w-z_2}{qw-z_2} - (q+1) + q\tfrac{z_2-qz_1}{qz_2-z_1}\tfrac{w-z_2}{qw-z_2} + q\tfrac{w-z_2}{qw-z_2} = 0$$

which is perhaps a fact about representations of $GL_2 \times GL_1 \times \mathbf{C}^*$, or the geometry of two planes in $\mathbf{C}^4$ meeting at 0, or even about commuting quasi-particles on a line, and which in any case is the "reason" that the Serre relations hold in our geometrically defined algebras, that Δ is a homomorphism, that $(,)$ is well defined...

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