RIEMANNIAN GEOMETRY. EXAMPLES 2.

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Let M^n be a Riemannian manifold and let $p \in M$. Given a unit norm vector $u \in T_pM$ show that the volume density $\theta(t,u)$ satisfies:

$$\theta(t,u) = t^{n-1} - \frac{(n-1)\,\mathrm{Ric}_p(u)}{6}\,t^{n+1} + O(t^{n+2}).$$

2. If q is a cut point of p along the geodesic γ, show that p is a cut point of q along the geodesic $-\gamma$.

3. Assume that the sectional curvatures of M are less than or equal to k. Show that for every $p \in M$ we have $V(p,r) \geq V_k(r)$ for all $r \leq \min\{\mathrm{inj}(p), \pi/\sqrt{k}\}$. Show that equality holds for some fixed r if and only if $B(p,r)$ is isometric to the ball of radius r in the constant curvature space form of curvature k. [*Hint: you may find this problem a bit difficult, see Theorem 3.7 in Chavel's book.*]

4. Let M be a Riemannian manifold let $p \in M$. Given a unit norm vector $u \in T_pM$ show that if $t \in [0, \mathrm{inj}(p))$ then $\theta(t,u) = \theta(t, -\dot{\gamma}(t))$, where $\gamma(t)$ is the geodesic with initial conditions (p,u).

5. (Due to E. Calabi and S.T. Yau.) Show that if M is complete non-compact, with nonnegative Ricci curvature, then for any $p \in M$ we have

$$V(p,r) \geq \mathrm{const}_p\, r$$

as $r \to +\infty$. (Hint: Pick a unit speed ray $\gamma : [0,+\infty) \to M$ from p to ∞ and let $x_k = \gamma(k)$. Show first that

$$V(x_k, k-1) \geq \left(\frac{k-1}{k+1}\right)^n V(x_k, k+1),$$

which implies (why?)

$$V(p, 2k) \geq V(x_k, k-1) \geq \frac{(k-1)^n}{(k+1)^n - (k-1)^n}\, V(p,1).)$$

Show that the result is sharp, i.e., exhibit a Riemannian manifold for which $V(p,r) \sim \mathrm{const}_p\, r$ as $r \to +\infty$.

6. Show that a complete Riemannian manifold M^n with nonnegative Ricci curvature such that

$$\lim_{r\to+\infty} \frac{V(p,r)}{w_n r^n} = 1$$

for some $p \in M$, must be isometric to the Euclidean space. (Here ω_n is the volume of the unit ball in the Euclidean n-space.)

7. Let M^n be a closed Riemannian manifold ($n \geq 2$). Show that M is simply connected if and only if the cut locus of any point is simply connected. (You may assume that the inclusion map $\iota : C_p \hookrightarrow M$ induces isomorphisms $\iota_* : \pi_i(C_p) \to \pi_i(M)$ for $1 \leq i \leq n-2$ and an epimorphism $\iota_* : \pi_{n-1}(C_p) \to \pi_{n-1}(M)$.)

8. Given $p \in M$, let C_p be the cut locus of p and Q_p^1 be the set of first conjugate points to p. For S^2 show that $C_p \cap Q_p^1$ is not empty for any point p and any metric on S^2. A. Weinstein has constructed on any Riemannian manifold other than S^2 a metric for which $C_p \cap Q_p^1$ is empty for some point p.

9. Let M be a closed Riemannian manifold. Show that M is simply connected if $Q_p^1 = C_p$.

10. A Riemannian manifold M is said to be *two-point homogeneous* if given $p, q \in M$ and unit vectors $u \in T_pM$, $v \in T_qM$, there exists an isometry f such that $f(p) = q$ and $df_q(u) = v$. Show that M is two-point homogeneous if and only if given p_1, p_2 and q_1, q_2 in M such that $d(p_1, p_2) = d(q_1, q_2)$ there exists an isometry f such that $f(p_i) = q_i$ for $i = 1, 2$.

11. A complete Riemannian manifold M is said to be *harmonic* if for any point $p \in M$, the volume density $\theta(t, u)$ is independent of the unit norm vector $u \in T_pM$, for all positive t. Show that a two-point homogeneous space is harmonic. Show that if M is harmonic then θ depends only on t and not on p. (The Lichnerowicz conjecture states that the universal covering of a harmonic manifold must be two-point homogeneous. Recently, Z. Szabo verified the conjecture for closed harmonic manifolds with finite fundamental group.)