THE PRESSURE OF RICCI CURVATURE

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Abstract. Let (M^n, g) be a closed Riemannian manifold and let κ_0 be any positive upper bound for the sectional curvature. We prove that

$$P\left(\frac{r_g}{2\sqrt{\kappa_0}}\right) \leq \frac{n-1}{2}\sqrt{\kappa_0},$$

where $P(f)$ stands for the topological pressure of a function f on the unit sphere bundle SM and $r_g(v)$ is the Ricci curvature in the direction of $v \in SM$. This result gives rise to several estimates for the various entropies of the geodesic flow which in turn have several consequences. One of them is entropy rigidity for those metrics in a hyperbolic manifold whose normalized total scalar curvature is bigger than that of the hyperbolic metric.

1. Introduction

Let (M^n, g) be a closed Riemannian manifold and let SM be the unit sphere bundle. Given a continuous function $f : SM \to \mathbb{R}$, let $P(f)$ be the topological pressure of the function f with respect to the geodesic flow Φ_t of g. The Ricci curvature of g can be regarded as a function $r_g : SM \to \mathbb{R}$ and it seems reasonable to think that its pressure should have some relevance in the study of the relationship between the dynamics of the geodesic flow of g and the geometry of g.

We recall the definition of pressure. Given $T > 0$ and a point $v \in SM$, set

$$f_T(v) := \int_0^T f(\Phi_t(v))\, dt.$$

We say that a set $E \subset SM$ is (T, ε)-separated if given $v_1 \neq v_2 \in E$, there exists $t \in [0, T]$ for which the distance between $\Phi_t(v_1)$ and $\Phi_t(v_2)$ is at least ε. Set,

$$q(T, \varepsilon, f) := \sup\left\{\sum_{v \in E} e^{f_T(v)} : \quad E \text{ is } (T, \varepsilon)\text{-separated}\right\},$$

$$q(\varepsilon, f) := \limsup_{T \to \infty} \frac{1}{T} \log q(T, \varepsilon, f).$$

The topological pressure is defined to be:

$$P(f) = \lim_{\varepsilon \to 0} q(\varepsilon, f).$$

G. P. Paternain was partially supported by CIMAT, Guanajuato, México.

J. Petean is supported by grant 37558-E of CONACYT.

The topological entropy h_{top} of Φ is $P(0)$. The variational principle for topological pressure says:

$$P(f) = \sup_{\mu \in \mathcal{M}(\Phi)} \left(h_\mu + \int_{SM} f \, d\mu \right),$$

where $\mathcal{M}(\Phi)$ is the set of all Φ-invariant Borel probability measures and h_μ is the entropy of the measure μ [11]. The study of the map $f \mapsto P(f)$ is important since it determines the members of $\mathcal{M}(\Phi)$ and when the entropy map $\mu \mapsto h_\mu$ is upper semi-continuous on $\mathcal{M}(\Phi)$ the knowledge of $P(f)$ for all f is equivalent to the knowledge of $\mathcal{M}(\Phi)$ and h_μ for all $\mu \in \mathcal{M}(\Phi)$ [11]. Also, the variational principle gives a natural way of selecting interesting members of $\mathcal{M}(\Phi)$.

In this note we show:

Theorem 1.1. *Let (M^n, g) be a closed Riemannian manifold. For any $v \in TM$ let $\lambda_1(v), ..., \lambda_{n-1}(v)$ be the eigenvalues of the curvature tensor at v. Let $E_g : SM \to \mathbb{R}$ be the function given by $E_g(v) = -\frac{1}{2}\sum_{i=1}^{n-1} |1 - \lambda_i(v)|$. Then*

$$P(E_g) \leq 0$$

and equality holds if g has constant curvature ± 1.

Observe that we can rephrase Theorem 1.1 by saying that for any Φ-invariant Borel probability measure μ,

$$h_\mu \leq \frac{1}{2} \int_{SM} \sum_{i=1}^{n-1} |1 - \lambda_i(v)| \, d\mu.$$

Corollary 1.2. *Let (M^n, g) be a closed Riemannian manifold and let κ_0 be any positive upper bound for the sectional curvature. Then*

$$P\left(\frac{r_g}{2\sqrt{\kappa_0}} \right) \leq \frac{n-1}{2} \sqrt{\kappa_0}.$$

We obtain interesting consequences by considering the Liouville measure μ_ℓ and a measure of maximal entropy μ_0 and applying the variational principle for the topological pressure.

Corollary 1.3. *Let (M^n, g) be a closed Riemannian manifold and let κ_0 be any positive upper bound for the sectional curvature. Then*

$$\begin{aligned} h_{top}(g) &\leq \frac{n-1}{2}\sqrt{\kappa_0} - \frac{\int_{SM} r_g \, d\mu_0}{2\sqrt{\kappa_0}} \\ &\leq \frac{n-1}{2}\sqrt{\kappa_0} - \frac{\min_{v \in SM} \, r_g(v)}{2\sqrt{\kappa_0}}, \end{aligned}$$

where $h_{top}(g)$ is the topological entropy of the geodesic flow of g.

The second upper bound for the topological entropy in the corollary was obtained in [9].

Corollary 1.4. *Let (M^n, g) be a closed Riemannian manifold and let κ_0 be any positive upper bound for the sectional curvature. Then*

$$h_{\mu_\ell}(g) + \frac{1}{2\, n\, V\, \sqrt{\kappa_0}} \int_M s_g(x)\, dx \leq \frac{n-1}{2}\sqrt{\kappa_0},$$

where $h_{\mu_\ell}(g)$ is the Liouville entropy of the geodesic flow, s_g is the scalar curvature, V is the volume of g and dx is the Riemannian volume element (not normalized).

Let k be a positive number such that $|K(P)| \leq k$ for all 2-planes P. Then, clearly $r_g \geq -(n-1)k\, g$ and hence Corollary 1.3 gives

$$h_{top}(g) \leq \frac{n-1}{2}\sqrt{k} + \frac{n-1}{2}\sqrt{k} = (n-1)\sqrt{k}.$$

The latter inequality, which is certainly weaker than that of Corollary 1.3, was first proved by A. Manning in [7].

The estimates of A. Freire and R. Mañé given in [4, Corollary II.1] can be derived from Corollary 1.2, at least in the non-positive curvature case (the proof is essentially the same as the one of Theorem 2.4 below and so we will omit it). The estimates say that if there are no conjugate points

$$h_\mu \leq \left(-(n-1) \int_M r_g\, d\mu \right)^{1/2},$$

where μ is any Φ-invariant Borel probability measure. Note that Freire and Mañé's convention for Ricci curvature differs from ours by a factor of $n-1$. Corollary 1.2 gives in fact the estimates of Freire and Mañé even when there are conjugate points, as long as

$$\int_M r_g\, d\mu \leq 0,$$

and g has sectional curvature bounded above by

$$-\frac{1}{n-1}\int_M r_g\, d\mu.$$

On the other hand, Freire and Mañé's proof works for an arbitrary manifold with no conjugate points without any further restrictions on curvature.

Finally we mention that in the case of non-positively curved manifolds there are lower bounds for the entropy in terms of curvature [1, 4, 8].

We will prove Theorem 1.1 and Corollary 1.2 in Section 3. Corollaries 1.3 and 1.4 follow directly from Corollary 1.2 and the variational principle for pressure. Section 2 contains several consequences of these corollaries.

Acknowledgements: We thank the referee for comments and suggestions for improvement. The first author thanks the Centro de Investigación en Matemática, Guanajuato, México for hospitality while this work was in progress.

2. Consequences of the estimates

Corollary 2.1. *Let M be a closed surface with Euler characteristic χ. Let g be any Riemannian metric whose curvature is bounded above by a^2. Then*

$$h_{\mu_\ell}(g) \leq \frac{a}{2} - \frac{\pi\chi}{aA},$$

where A is the area of (M, g).

Proof. By the Gauss-Bonnet theorem

$$\int_M s_g(x)\, dx = 4\pi\chi$$

and the claim follows immediately from Corollary 1.4.

□

In [5] A. Katok proved that for the geodesic flow of a unit-area Riemannian metric *without focal points* on a surface of negative Euler characteristic χ the Liouville and topological entropies lie on either side of $\sqrt{-2\pi\chi}$, with equality (on either side) only for constant curvature metrics. In fact, he showed a similar statement in any dimension provided that the metric is in the conformal class of a locally symmetric metric, but always assuming it does not have focal points. Katok [5, p. 347] conjectured that Liouville measure has maximal entropy only for locally symmetric spaces. This conjecture has generated enormous amounts of work and remains unsolved. L. Flaminio [3] has proved that the conjecture holds in a neighborhood of a hyperbolic metric and that in dimension 3 is no longer the case that a hyperbolic metric with unit volume minimizes Liouville entropy.

The next theorem verifies Katok's conjecture for a large class of metrics in a hyperbolic manifold.

Theorem 2.2. *Let M^n be a closed manifold that admits a metric g_0 of constant negative curvature that we normalize to have volume one and let $-a^2$ be its curvature. Let g be any Riemannian metric with unit volume and sectional curvature bounded from above by a^2. Assume that*

$$\int_M s_g(x)\, dx \geq -n(n-1)a^2.$$

Then $h_{\mu_\ell}(g) = h_{top}(g)$ if and only if g has constant sectional curvature.

Proof. On account of the results proved by G. Besson, G. Courtois and S. Gallot in [2] we have

$$h_{top}(g) \geq (n-1)a$$

with equality if and only if g has constant sectional curvature. If $h_{\mu_\ell}(g) = h_{top}(g)$ Corollary 1.4 implies

$$(n-1)a + \frac{1}{2n\,a}\int_M s_g(x)\, dx \leq h_{\mu_\ell}(g) + \frac{1}{2n\,a}\int_M s_g(x)\, dx \leq \frac{(n-1)a}{2}$$

and thus

$$\int_M s_g(x)\,dx \le -n(n-1)a^2.$$

The hypothesis implies that equality must hold and hence $h_{top}(g) = (n-1)a$. Consequently g must have constant sectional curvature.

□

Using the theorem and some analysis of the total scalar curvature functional it is possible to prove Katok's conjecture for neighborhoods of metrics in the conformal class of g_0. More precisely we have:

Corollary 2.3. *Let M^n be a closed manifold that admits a metric g_0 of constant negative curvature that we normalize to have volume one and let $-a^2$ be its curvature. Let g be any Riemannian metric in the conformal class of g_0 with unit volume and sectional curvature bounded from above strictly by a^2. Then there exists a neighborhood $\mathcal{U}$ of g with the following property: for any $g^* \in \mathcal{U}$ we have $h_{\mu_\ell}(g^*) = h_{top}(g^*)$ if and only if g^* has constant sectional curvature.*

We remark that this corollary contains new information even for surfaces, since our metrics are allowed to have conjugate points as long as their curvature is bounded from above by a^2.

Proof. When g is isometric to g_0 the statement of the corollary is proved in [3]. A metric of constant negative scalar curvature is the unique minimum of the total scalar curvature in its conformal class (with fixed volume, of course). Then if g is not isometric to g_0 we have:

$$\int_M s_g(x)\,dx > -n(n-1)a^2.$$

Hence there exists a neighborhood $\mathcal{U}$ of g such that any $g^* \in \mathcal{U}$ has sectional curvature bounded from above by a^2 and

$$\int_M s_{g^*}(x)\,dx > -n(n-1)a^2.$$

The corollary now follows from the theorem. □

We show now that the locally symmetric metric on a hyperbolic manifold achieves the unique maximum (for metrics of non-positive curvature) of the functional introduced by G. Knieper in [6]. It is clear that the maximum over all the metrics on the manifold cannot be achieved, and so some restriction on the metrics considered must be imposed. Our estimates show that the hyperbolic metric is a maximum on the space of metrics of curvature bounded by some natural positive constant, but we cannot prove uniqueness in this case. It is not hard to see that the results in the case of non-positive curvature can also be derived from the results of Freire and Mañé [4].

Theorem 2.4. *Let M^n ($n \ge 3$) be a closed manifold that admits a metric g_0 of constant negative curvature that we normalize to have volume one and let $-a^2$ be its* *curvature. Let $\mathcal{R}$ be the space of all non-positively curved metrics with volume one. Then the functional*

$$\mathcal{R} \ni g \mapsto F(g) := h_{\mu_\ell}(g) + \frac{1}{2n\,a}\int_M s_g(x)\,dx,$$

achieves a unique (up to isometry) absolute maximum at the metric g_0. The metric g_0 is actually an absolute maximum over the space of metrics with unit volume and curvature bounded above by a^2.

Proof. It is well known that $h_{\mu_\ell}(g_0) = (n-1)a$. Then the fact that g_0 gives an absolute maximum follows directly from Corollary 1.4 by picking $\kappa_0 = a^2$.

Now consider any metric $g \in \mathcal{R}$. Since the function

$$(0,\infty) \ni k \mapsto \frac{(n-1)k}{2} - \frac{1}{2n\,k}\int_M s_g(x)\,dx$$

achieves its minimum at

$$k = \left(-\frac{1}{n(n-1)}\int_M s_g(x)\,dx\right)^{1/2},$$

we obtain that for all k:

$$h_{\mu_\ell}(g) \le \left(-\frac{n-1}{n}\int_M s_g(x)\,dx\right)^{1/2} \le \frac{(n-1)k}{2} - \frac{1}{2n\,k}\int_M s_g(x)\,dx.$$

Now suppose that g is a metric such that $F(g) = \frac{(n-1)a}{2}$. Then, by the previous inequality, we have that

$$h_{\mu_\ell}(g) = \left(-\frac{n-1}{n}\int_M s_g(x)\,dx\right)^{1/2}.$$

Freire and Mañé proved in [4] that this equality implies that g itself must have constant negative curvature and the theorem now follows from Mostow rigidity. □

Remark 2.5. Note that the theorem can be sharpened slightly by letting $\mathcal{R}$ be the space of all unit volume metrics with no conjugate points and sectional curvature bounded above by $-\frac{1}{n(n-1)}\int_M s_g(x)\,dx$.

When $n = 2$ we loose uniqueness but it is still true that any absolute maximum must have constant sectional curvature.

Remark 2.6. Observe that a similar result can also be obtained for the functional $g \mapsto P(r_g/2a)$.

3. Proof of Theorem 1.1 and Corollary 1.2

Let $SM \subset TM$ be the unit sphere bundle of M. We will consider the geodesic flow Φ_t of g acting on SM.

The metric g on M induces a metric on TM (the Sasaki metric) and for any $\theta \in TM$ an orthogonal decomposition of $T_\theta TM$ into horizontal and vertical parts:

$T_\theta TM = H_\theta \oplus V_\theta$. The differential of Φ_t has a nice expression in geometric terms; given $\xi = (w_1, w_2) \in T_\theta TM$, $d_\theta \Phi_t(\xi) = (J_\xi(t), \dot{J}_\xi(t))$, where J_ξ is the Jacobi field with initial conditions $J_\xi(0) = w_1$ and $\dot{J}_\xi(0) = w_2$.

If $\theta = (x, v)$ we will denote by $S(\theta)$ the orthogonal complement of $(v, 0)$ in $T_\theta SM$. The subspaces $S(\theta)$ are invariant through the differential of the geodesic flow and we will consider $d_\theta \Phi_t$ as a map between $S(\theta)$ and $S(\Phi_t(\theta))$.

The proof of the theorem will be based on an inequality which we will describe now.

Given two real vector spaces with inner product V and W of the same dimension and a linear transformation $f : V \to W$ the *expansion* of f, $\mathrm{ex}(f)$, is the supremum over all non-trivial subspaces of V of the absolute value of the determinant of $f|_V$. An important (and obvious) property of the expansion is that given two linear maps f and g we have $\mathrm{ex}(f\,g) \le \mathrm{ex}(f)\,\mathrm{ex}(g)$. This will be used below.

Lemma 3.1. *Let $f : X \to X$ be a diffeomorphism of class C^1 and let μ be any invariant Borel probability measure. Then*

$$h_\mu(f) \le \int_X \log \mathrm{ex}(d_x f)\; d\mu(x).$$

Proof. Let $\chi_+(x)$ be the sum of the positive Liapunov exponents. By Ruelle's inequality [10]:

$$h_\mu(f) \le \int_X \chi_+(x)\; d\mu(x).$$

On the other hand it is well known that for a.e. x [10] we have,

$$\chi_+(x) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{ex}(d_x f^n).$$

Observe that the sequence $a_n(x) := \log \mathrm{ex}(d_x f^n)$ is subadditive, i.e.,

$$a_{n+m}(x) \le a_n(x) + a_m(f^n(x))$$

and by Kingman's subadditive ergodic theorem we get that for a.e. x

$$\lim_{n\to\infty} \frac{1}{n} a_n(x) = \inf_n \frac{1}{n} \widetilde{a}_n$$

where $\widetilde{a}_n$ denotes the Birkhoff mean of a_n. Thus

$$\int_X \lim_{n\to\infty} \frac{1}{n} a_n(x)\; d\mu(x) = \int_X \inf_n \frac{1}{n} \widetilde{a}_n\; d\mu(x)$$

and by Birkhoff's ergodic theorem:

$$\int_X \lim_{n\to\infty} \frac{1}{n} \log \mathrm{ex}(d_x f^n)\; d\mu(x) \le \int_X \log \mathrm{ex}(d_x f)\; d\mu(x).$$

□

Let us apply the lemma to $f = \Phi_\delta$ for small $\delta > 0$. Then for any invariant Borel probability measure μ we obtain:

$$h_\mu(g) \leq \liminf_{\delta\to 0} \frac{1}{\delta}\int_{SM} \log \operatorname{ex}(d_\theta \Phi_\delta)\ d\mu(\theta).$$

Below we will show:

Lemma 3.2.

$$\lim_{\delta\to 0} \frac{1}{\delta} \log \operatorname{ex}(d_\theta \Phi_\delta) = \frac{1}{2}\sum_{i=1}^{n-1} |\lambda_i(\theta) - 1|.$$

Since the function $\frac{1}{\delta}\log \operatorname{ex}(d_\theta\Phi_\delta)$ is uniformly bounded from above (SM is compact) we can apply Lebesgue's dominated convergence theorem to obtain:

$$\begin{aligned} h_\mu(g) &\leq \int_{SM} \lim_{\delta\to 0}\frac{1}{\delta}\log \operatorname{ex}(d_\theta\Phi_\delta)\ d\mu(\theta) \\ &= \int_{SM} \frac{1}{2}\sum_{i=1}^{n-1}|\lambda_i(\theta)-1|\, d\mu(\theta) \end{aligned}$$

and Theorem 1.1 follows from the variational principle for the topological pressure.

Proof of Lemma 3.2. We will study $\operatorname{ex}(d_\theta\Phi_\delta)$ for small values of δ. Consider the polar decomposition of $d_\theta\Phi_t$: $d_\theta\Phi_t = O_t(\theta)\ L_t(\theta)$ where $O_t(\theta) : S(\theta) \to S(\Phi_t(\theta))$ is a linear isometry and $L_t(\theta)$ is a symmetric positive endomorphism of $S(\theta)$. Then $\operatorname{ex}(d\Phi_t) = \operatorname{ex}(L_t)$. Of course, L_t can be given explicitly: $L_t = ((d\Phi_t)^*\ (d\Phi_t))^{1/2}$.

Consider the map $\mathcal{R} : S(\theta) \to S(\theta)$ which in the decomposition into horizontal and vertical parts is given by $\mathcal{R}(w_1, w_2) = (w_2, -R(v, w_1)v)$, where R is the curvature tensor and $\theta = (x, v) \in SM$. We will need the following lemma proved in [9, Lemma 3.2].

Lemma 3.3. $((d\Phi_\delta)^*\ (d\Phi_\delta))^{1/2} = Id + \frac{\delta}{2}(\mathcal{R} + \mathcal{R}^*) + O(\delta^2)$.

For the convenience of the reader we include a brief sketch of the proof. Let $\xi = (w_1, w_2) \in S(\theta)$. Use parallel translation to identify vectors at one point along the geodesic defined by θ with vectors at another point. With this understanding, we get using the Jacobi equation that

$$\begin{aligned} d\Phi_\delta(w_1, w_2) &= (J_\xi(\delta), \dot{J}_\xi(\delta)) \\ &= (J_\xi(0) + \delta\dot{J}_\xi(0), \dot{J}_\xi(0) + \delta\ddot{J}_\xi(0)) + O(\delta^2) \\ &= (J_\xi(0) + \delta\dot{J}_\xi(0), \dot{J}_\xi(0) - R(v, J_\xi(0))v) + O(\delta^2) \\ &= (Id + \mathcal{R})(w_1, w_2) + O(\delta^2). \end{aligned}$$

The lemma follows easily.

From the lemma we obtain that $\mathrm{ex}(d\Phi_\delta) = \mathrm{ex}\left(Id + \frac{\delta}{2}(\mathcal{R}+\mathcal{R}^*)\right) + O(\delta^2)$; and we are left to compute $\mathrm{ex}(Id+\frac{\delta}{2}(\mathcal{R}+\mathcal{R}^*))$. Since $Id+\frac{\delta}{2}(\mathcal{R}+\mathcal{R}^*)$ is a positive definite symmetric endomorphism of $S(\theta)$, its expansion is the product of the eigenvalues which are greater than or equal to one, provided that there exists at least one eigenvalue greater than or equal to one.

We can compute the eigenvalues explicitly using the orthonormal basis of $S(\theta)$ given by

$$\{(e_1,e_1),\ldots,(e_{n-1},e_{n-1}),(e_1,-e_1),\ldots,(e_{n-1},-e_{n-1})\},$$

where $\{e_1,...,e_{n-1}\}$ is an orthonormal basis of eigenvectors for the symmetric linear map $w \mapsto R(v,w)v$ on $v^\perp$.

Suppose that $R(v,e_i)v = \lambda_i e_i$. Then it is easy to check that

$$(Id + (\delta/2)(\mathcal{R}+\mathcal{R}^*))\,(e_i,e_i) = (1+(\delta/2)(1-\lambda_i))\,(e_i,e_i)$$

and

$$(Id + (\delta/2)(\mathcal{R}+\mathcal{R}^*))\,(e_i,-e_i) = (1+(\delta/2)(\lambda_i-1))\,(e_i,-e_i).$$

Therefore,

$$\mathrm{ex}(d\Phi_\delta) = \prod_{\lambda_i\le 1}\left(1+\frac{\delta}{2}(1-\lambda_i)\right)\prod_{\lambda_i>1}\left(1+\frac{\delta}{2}(\lambda_i-1)\right)+O(\delta^2).$$

Hence

$$\mathrm{ex}(d\Phi_\delta) = \prod_{i=1}^{n-1}\left(1+\frac{\delta}{2}|\lambda_i-1|\right)+O(\delta^2) = 1 \;+\; \frac{\delta}{2}\sum_{i=1}^{n-1}|\lambda_i-1| \;+\; O(\delta^2).$$

□

Corollary 1.2 follows easily: if $\lambda_i \le 1$ for all i we get that

$$P\left(-\frac{1}{2}(n-1-r_g(v))\right) = P\left(\frac{r_g}{2}\right) - \frac{n-1}{2} \le 0,$$

where $r_g(v)$ denotes, as before, the Ricci curvature in the direction of v.

Therefore if the sectional curvature of g is bounded above by 1 we get, from the previous discussion, that

$$P\left(\frac{r_g}{2}\right) \le \frac{n-1}{2}.$$

For a general g, let k be any positive upper bound for the sectional curvature. The metric $g_k = kg$ has sectional curvature bounded above by one and from the previous observation we get that

$$P\left(\frac{r_{g_k}}{2}\right) \le \frac{n-1}{2}.$$

But $P\left(\frac{r_{g_k}}{2}\right) = \frac{1}{\sqrt{k}} P\left(\frac{r_g}{2\sqrt{k}}\right)$ and therefore we get

$$P\left(\frac{r_g}{2\sqrt{k}}\right) \leq \frac{n-1}{2}\sqrt{k},$$

as desired.

□

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