GLOBAL RIEMANNIAN GEOMETRY. EXAMPLES 1.

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Let G be a Lie group endowed with a Riemannian metric which is left and right invariant and let X, Y, Z be left invariant vector fields of G.

(a) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$.

(b) Prove that $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$.

(c) Suppose that X and Y are orthonormal, and let $K(\sigma)$ be the sectional curvature of the 2-plane σ spanned by X and Y. Prove that

$$K(\sigma) = \frac{1}{4}||[X, Y]||^2.$$

2. Let M be a Riemannian manifold. M is said to be *locally symmetric* if $\nabla R = 0$, where R is the curvature tensor of M.

(a) Let M be a locally symmetric space and let $\gamma : [0, \ell] \to M$ be a geodesic of M. Let X, Y, Z be parallel vector fields along γ. Prove that $R(X, Y)Z$ is a parallel field along γ.

(b) Suppose that M is locally symmetric, connected and 2-dimensional. Prove that M has constant sectional curvature.

(c) Prove that if M has constant sectional curvature, then it is locally symmetric.

3. Prove that the scalar curvature $s(p)$, $p \in M$ is given by

$$s(p) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} Ric_p(x)\, dx,$$

where ω_{n-1} is the volume of the unit sphere S^{n-1} in T_pM.

4. Let M be a Riemannian manifold, $\gamma : [0, 1] \to M$ a geodesic and J a Jacobi field along γ. Prove that there exists a parametrized surface $f(t, s)$ such that $f(0, t) = \gamma(t)$, the curves $t \mapsto f(s, t)$ are geodesics and $J(t) = \frac{\partial f}{\partial s}(t, 0)$.

5. Let $\gamma : [0, \infty) \to M$ be a geodesic in a locally symmetric space (cf. Problem 2) and let $(p, v) = (\gamma(0), \dot\gamma(0))$. Consider the linear map $K_v : T_pM \to T_pM$ given by $K_v(x) = R(v, x)v$, $x \in T_pM$.

(a) Prove that K_v is self-adjoint.

(b) Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM which diagonalises K_v, i.e., $K_v(e_i) = \lambda_i\, e_i$, for $i = 1, \ldots, n$. Consider $e_i(t)$, the parallel transport of e_i along γ. Show that for all t, $K_{\dot\gamma(t)}(e_i(t)) = \lambda_i\, e_i(t)$, where λ_i is independent of t.

(c) Solve the Jacobi equation and show that the conjugate points to p along γ are given by $\gamma(\pi\, k/\sqrt{\lambda_i})$, where k is a positive integer and λ_i is a positive eigenvalue of K_v.

6. Let M be a Riemannian manifold of dimension 2, i.e., a surface. Let $B_\delta(p)$ be a normal ball around $p \in M$. Let

$$f(\rho, \theta) = \exp_p(\rho\, v(\theta)), \quad 0 < \rho < \delta, \quad -\pi < \theta < \pi,$$

where $\theta \mapsto v(\theta)$ describes a circle of radius δ in T_pM.

(a) Show that the coefficients g_{ij} of the Riemannian metric in the coordinates (ρ, θ) are given by:

$$g_{12} = 0, \quad g_{11} = \left|\frac{\partial f}{\partial \rho}\right|^2 = |v(\theta)|^2 = 1, \quad g_{22} = \left|\frac{\partial f}{\partial \theta}\right|^2.$$

(b) Show that along the geodesic $f(\rho, 0)$ we have

$$(\sqrt{g_{22}})_{\rho\rho} = -K(p)\,\rho + r(\rho),$$

where $\lim_{\rho\to 0} r(\rho)/\rho = 0$ and $K(p)$ is the sectional curvature of M at p.

(c) Prove that

$$\lim_{\rho\to 0} \frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = -K(p).$$

The last expression is the value of the Gaussian curvature of M in polar coordinates. Assuming this fact from the theory of surfaces, this problem shows that in dimension 2, the sectional curvature coincides with the Gaussian curvature.

7. Let N be a Riemannian manifold and let $f : M \to N$ be a local diffeomorphism. Show that one can put a Riemannian metric on M such that f becomes a local isometry. Show that if M is complete then N is complete. Is the converse true? Is the converse true if f is a covering map?

8. A geodesic $\gamma : [0, \infty) \to M$ is called a *ray* if it minimizes the distance between $\gamma(0)$ and $\gamma(s)$ for all $s \in (0, \infty)$. Show that if M is complete and non-compact, there is a ray leaving from every point in M.

9. A Riemmanian manifold M is said to be *homogeneous* if given p and q in M, there exists an isometry of M taking p to q. Show that a homogeneous Riemannian manifold is complete.

10. Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : \quad |z_1|^2 + |z_2|^2 = 1\}$ and let $h : S^3 \mapsto S^3$ be given by

$$h(z_1, z_2) = (e^{2\,\pi\,i/q}\, z_1, e^{2\,\pi\,i\,r/q}\, z_2),$$

where q and r are coprime integers and $q > 1$.

(a) Show that $G = \{id, h, \ldots, h^{q-1}\}$ is a group of isometries of S^3 (equipped with its standard metric) that acts in such a way that S^3/G is a manifold and the projection $p : S^3 \to S^3/G$ is a local diffeomorphism. (The manifolds S^3/G are called *lens spaces*.)

(b) Consider in S^3/G the metric induced by p. Show that all the geodesics of S^3/G are closed, but they could have different lengths.

11. Let M be a complete Riemannian manifold and let $N \subset M$ be a closed submanifold. Let $p \in M$, $p \notin N$, and let $d(p, N)$ be the distance from p to N. Show that there exists a point $q \in N$ such that $d(p, q) = d(p, N)$. Show that a minimizing geodesic between p and q must be orthogonal to N at q.

12. Let M be an orientable Riemannian manifold of even dimension and positive sectional curvature. Show that any closed geodesic γ in M is homotopic to a closed curve with length strictly smaller than that of γ.

13. Suppose that for every smooth Riemannian metric on a manifold M, M is complete. Show that M is compact (Hint: think about rays as in Problem 8).