

## GROUPS EXAMPLES 1

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The questions on this sheet are not all equally difficult and the harder ones are marked with \*'s. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at [g.p.paternain@dpms.cam.ac.uk](mailto:g.p.paternain@dpms.cam.ac.uk).

1. Let  $G$  be any group. Show that the identity  $e$  is the unique solution of the equation  $x^2 = x$ .
2. Let  $G$  be any group. Given  $g \in G$ , show by induction that  $g^{m+n} = g^m g^n$  for all integers  $m$  and  $n$ .
3. Let  $G$  be a finite group. Show that there exists a positive integer  $n$  such that  $g^n = e$  for all  $g \in G$  (the  $n$  that you find must be independent of  $g$ ).
4. Let  $G = \{x \in \mathbb{R} : x \neq -1\}$ , where  $\mathbb{R}$  is the set of real numbers, and let  $x * y = x + y + xy$ , where  $xy$  denotes the usual product of two real numbers. Show that  $(G, *)$  is a group. What is the inverse  $2^{-1}$  of 2 in this group? Solve the equation  $2 * x * 5 = 6$ .
5. Let  $S$  be a finite subgroup of the multiplicative group of non-zero complex numbers. Show that for some positive integer  $n$ ,  $S$  is exactly the group of  $n$ -th roots of unity.
6. Let  $\theta : G \rightarrow K$  be a homomorphism between finite groups. Given  $g \in G$ , show that the order of  $\theta(g)$  must divide the order of  $g$ . Describe all the homomorphisms from  $C_n$  to  $C_m$ , in particular, what happens if  $n$  and  $m$  are coprime?
7. Let  $C_n$  be the cyclic group with  $n$  elements and  $D_{2n}$  the group of symmetries of the regular  $n$ -gon. If  $n$  is odd and  $\theta : D_{2n} \rightarrow C_n$  is a homomorphism, show that  $\theta(g) = e$  for all  $g \in D_{2n}$ . What can you say if  $n$  is even?
8. Consider the Möbius maps  $f(z) = e^{2\pi i/n} z$  and  $g(z) = 1/z$ . Show that the subgroup  $G$  of the Möbius group  $\mathcal{M}$  generated by  $f$  and  $g$  is a dihedral group. [“Generated” here means that every element in  $G$  is the product of elements of the form  $f, g, f^{-1}$  and  $g^{-1}$ .]
9. Express the Möbius transformation  $f(z) = (2z + 3)/(z - 4)$  as the composition of maps of the form  $z \mapsto az$ ,  $z \mapsto z + b$  and  $z \mapsto 1/z$ . Hence show that  $f$  maps the circle  $|z - 2i| = 2$  onto the circle  $|8z + (6 + 11i)| = 11$ .
10. Let  $G$  be the subgroup of Möbius transformations which map the set  $\{0, 1, \infty\}$  onto itself. Find all the elements in  $G$ . To which standard group is  $G$  isomorphic? Find the group of Möbius transformations which map the set  $\{0, 2, \infty\}$  onto itself.
11. Show that a subgroup of a cyclic group is cyclic.
12. Let  $G$  be a group in which every element other than the identity has order two. Show that  $G$  is abelian.
13. Let  $G$  be a group of even order. Show that  $G$  contains an element of order two.
- 14\*. Describe the finite subgroups of the group of isometries of the plane. [If we think of the plane as  $\mathbb{C}$  you may assume that all isometries have the form  $z \mapsto az + b$  or  $z \mapsto a\bar{z} + b$ , where  $a$  and  $b$  are complex numbers and in both cases  $|a| = 1$ . Hint: is there a point in the plane fixed by all the elements in the group? ]