GROUPS EXAMPLES 1

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The questions on this sheet are not all equally difficult and the harder ones are marked with $*$'s. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Let G be any group. Show that identity e is the unique solution of the equation $x^2 = x$.

2. Let G be any group. Given $g \in G$, show that $g^{m+n} = g^m g^n$ for all integers m and n.

3. Let G be a finite group. Show that there exists a positive integer n such that $g^n = e$ for all $g \in G$.

4. Let $G = \{x \in \mathbb{R} : x \neq -1\}$, where $\mathbb{R}$ is the set of real numbers, and let $x * y = x + y + xy$, where xy denotes the usual product of two real numbers. Show that $(G, *)$ is a group. What is the inverse 2^{-1} of 2 in this group? Solve the equation $2 * x * 5 = 6$.

5. Let S be a finite subgroup of the multiplicative group of non-zero complex numbers. Show that for some positive integer n, S is exactly the group of n-th roots of unity.

6. Let C_n be the cyclic group with n elements and D_{2n} the group of symmetries of the regular n-gon. If n is odd and $\theta : D_{2n} \to C_n$ is a homomorphism, show that $\theta(g) = e$ for all $g \in D_{2n}$. What can you say if n is even? Find all the homomorphisms from C_n to C_m.

7. The surface of a torus (a doughnut ring) can be obtained by rotating a circle in $\mathbb{R}^3$ about a line in a plane of the circle (and not meeting the circle). It follows that points in the torus can be parametrised by two coordinates $(e^{i\theta}, e^{i\phi})$. Explain how to make the torus into a group.

8. Consider the Möbius maps $f(z) = e^{2\pi i/n} z$ and $g(z) = 1/z$. Show that the subgroup G of the Möbius group $\mathcal{M}$ generated by f and g is a dihedral group. ["Generated" here means that every element in G is the product of elements of the form f, g, f^{-1} and g^{-1}.]

9. Express the Möbius tranformation $f(z) = (2z + 3)/(z - 4)$ as the composition of maps of the form $z \mapsto az$, $z \mapsto z + b$ and $z \mapsto 1/z$. Hence show that f maps the circle $|z - 2i| = 2$ onto the circle $|8z + (6 + 11i)| = 11$.

10. Let G be the subgroup of Möbius transformations which map the set $\{0, 1, \infty\}$ onto itself. Find all the elements in G. To which standard group is G isomorphic? Find the group of Möbius transformations which map the set $\{0, 2, \infty\}$ onto itself.

11. Show that a subgroup of a cyclic group is cyclic.

12. Let G be a group in which every element other than the identity has order two. Show that G is abelian.

13. Let G be a group of even order. Show that G contains an element of order two.

14*. Describe the finite subgroups of the group of isometries of the plane. [If we think of the plane as $\mathbb{C}$ you may assume that all isometries have the form $z \mapsto az + b$ or $z \mapsto a\bar{z} + b$, where a and b are complex numbers and in both cases $|a| = 1$.]