DIFFERENTIAL GEOMETRY, PART III, EXAMPLES 2.

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`. Most of the examples in this sheet are taken from Alexei Kovalev's example sheets. The questions are not equally difficult. Those marked with * are not always harder, but are less central to the lectured material and may be regarded as a supplement for the enthusiasts.

1. Let W be a compact domain with boundary $X := \partial W$ in an oriented manifold. Suppose Y is an oriented manifold with $\dim Y = \dim X = n$. Let $f : X \to Y$ be a smooth map that extends smoothly to all W. Show that if ω is any n-form on Y then

$$\int_X f^*\omega = 0.$$

2. Let $f, g : X \to Y$ be smoothly homotopic maps between compact oriented n-manifolds. Show that for every n-form ω in Y,

$$\int_X f^*\omega = \int_X g^*\omega.$$

Assuming that the n-th de Rham group of a compact orientable n-manifold is $\mathbb{R}$, explain how to attach a number to any smooth map $f : X \to Y$ which only depends on the homotopy class of f and has the value 1 on orientation preserving diffeomorphisms. This number is called the *degree* of f and it can shown to be an integer.

Let S be a compact oriented surface in $\mathbb{R}^3$. Combine the above with the Gauss-Bonnet theorem to relate the degree of the Gauss map with a well known topological invariant of S.

3. Prove that a principal G-bundle $P \to B$ has a smooth global section if and only if P is a trivial bundle.

Let $\pi : P \to B$ be a principal G-bundle. Show that $\pi^*P \to P$ is a trivial principal G-bundle.

4. Show that map $[x_0 : x_1 : x_2 : x_3] \in \mathbb{R}P^3 \to [(x_0+ix_1) : (x_2+ix_3)] \in \mathbb{C}P^1$ defines a principal $U(1)$-bundle, the two standard coordinate patches on $\mathbb{C}P^1$ may be taken as trivializing neighbourhoods, and the transition function then is given by $\psi([z : 1]) = (z/|z|)^2$.

5. Let G be a matrix Lie group and X_i, $i = 1, \dots, d = \dim G$, a system of linearly independent left-invariant vector fields on G induced by a basis of $T_I G$. Show that the condition that $\omega^i(X_j) = \delta^i_j$ identically on G defines a system of linearly independent smooth 1-forms ω^i on G. Show further that the 1-forms ω^i are *left-invariant* in the sense that

$$L_g^*(\omega^i) = \omega^i, \qquad \text{for every } g \in G.$$

Let C^k_{ij} be a set of real constants determined by $[X_i, X_j] = C^k_{ij} X_k$. Deduce from the identity of Question 12 of Example Sheet 1 the formula $d\omega^k = -C^k_{ij}\omega^i \wedge \omega^j$.

6. Modify the construction of the Hopf bundle given in Lectures, replacing $\mathbb{C}$ everywhere by $\mathbb{R}$ to obtain a rank one real vector bundle over S^1. The total space of this $\mathbb{R}$-analogue of the Hopf (vector) bundle is thus a surface (2-dimensional manifold). Can you identify this surface?

7. Show that every (real) vector bundle can be given a positive definite inner product, varying smoothly with the fibres, i.e. given in each local trivialization (U_α, Φ_α) by a smooth map $g_\alpha : x \in$

$U_\alpha \to g_\alpha(x) \in \mathrm{Sym}_+(k, \mathbb{R})$. Here $k = \mathrm{rk}\, E$ and $\mathrm{Sym}_+(k, \mathbb{R})$ denotes the set of all real positive-definite $k \times k$ symmetric matrices.
[Hint: you might like to use a partition of unity.]

Deduce that any vector bundle admits an $O(n)$-structure. Identify the total space of the associated principal $O(n)$-bundle as a subset of E.

State the analogous results for complex vector bundles.

8. (i) Consider S^2 endowed with a Riemannian metric. Let Y denote the subset of TS^2 consisting of all the tangent vectors of unit length. Show that the tangent bundle TS^2 induces on Y a structure of a principal S^1-bundle over S^2.

(ii)* Show further that the 3-dimensional manifold Y is diffeomorphic to $\mathbb{R}P^3$.
[Hint for (ii): Find a correspondence between points of Y and *ordered pairs* of orthogonal unit vectors in $\mathbb{R}^3$. Recall also that $\mathbb{R}P^3$ is diffeomorphic to $SO(3)$.]

9. Let $E \to B$ be a rank 1 complex vector bundle with a unitary structure and A a unitary connection on E and $F(A)$ the curvature of A. Prove that the 2-form $\frac{1}{2\pi i}F(A)$ on B is closed, and its de Rham cohomology class is well-defined independent of the choice of A. (This class therefore depends only on the bundle E; it is called the *first Chern class of* E, denoted $c_1(E)$.)

10. Consider the situation as in the previous question but now suppose in addition that the base B of E is a compact oriented 2-dimensional manifold. Show that the *first Chern number of* E, $\int_B c_1(E) = \frac{1}{2\pi i}\int_B F(A)$, is well-defined (independent of the choice of connection A). Find the first Chern number of the Hopf vector bundle over $\mathbb{C}P^1$.

[You will need to consider a unitary structure on the Hopf bundle, as given in the Lectures, and construct a unitary connection. If the curvature form of this connection vanishes on some open neighbourhood in $\mathbb{C}P^1$ then the integral can be effectively computed over an open domain in $\mathbb{R}^2$, or $\mathbb{C}$, using Green's formula. Answer: -1.]

11.* Prove the following *integrability theorem* for flat connections. If E is a vector bundle over the open hypercube $H = \{x \in \mathbb{R}^n : \max_i |x_i| < 1\}$ and A is a flat connection on E then there is a bundle isomorphism taking E to the trivial bundle over H and A to the trivial (product) connection.
[Hint: it is a good idea to use induction in n. Killing the coefficient of A at any dx^k amounts to solving a linear ODE (with parameters).

12. (i) Assuming the integrability theorem stated in the previous question, deduce that if a vector bundle admits a flat connection then there is a choice of local trivializations of this bundle, so that the corresponding transition functions are *constant*, $\psi_{\beta\alpha}(x) \equiv h_{\beta\alpha}$, for all $x \in U_\beta \cap U_\alpha$.

(ii) Show further that a flat connection on a vector bundle over a simply-connected base manifold, B say, determines an isomorphism of this bundle to a trivial bundle, i.e. a (global) trivialization over all of B.
[Hint: covariant-constant sections.]

13. Verify that if M is a submanifold of $\mathbb{R}^N$ then the Euclidean inner product restricts to define a Riemannian metric on M.
Let the symbol dS^2 denote the expression for the induced metric on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Show that the Euclidean metric on $\mathbb{R}^n \setminus \{0\}$ can be expressed as $g = dr^2 + r^2 dS^2$, where $r = |x|$, $x \in \mathbb{R}^n$.
[You might like to consider the dimensions $n = 2$ or 3 first, using polar coordinates.]