DIFFERENTIAL GEOMETRY, PART III, EXAMPLES 1.

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`. Most of the examples in this sheet are taken from Alexei Kovalev's example sheets. The questions are not equally difficult. Those marked with * are not always harder, but are less central to the lectured material and may be regarded as a supplement for the enthusiasts.

1. Do the charts $\phi_1(x) = x$ and $\phi_2(x) = x^3$ $(x \in \mathbb{R})$ belong to the same C^∞ differentiable structure on $\mathbb{R}$?

Let R_j, $j = 1, 2$, be the manifold defined by using the chart ϕ_j on the topological space $\mathbb{R}$. Are R_1 and R_2 diffeomorphic?

Show that the subset $X = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ of the coordinate plane, with the induced topology from $\mathbb{R}^2$, does not admit any C^∞ differentiable structure and thus cannot be made into a manifold.

2. Show that the following groups are Lie groups (in particular, smooth manifolds):

(1) special linear group $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det A = 1\}$;
(2) unitary group $U(n) = \{A \in GL(n, \mathbb{C}) : AA^* = I\}$, where A^* denotes the conjugate transpose of A and I is the $n \times n$ identity matrix;
(3) special unitary group $SU(n) = \{A \in U(n) : \det A = 1\}$.
(4) $Sp(m) = \{A \in U(2m) : AJA^t = J\}$, where A^t denotes the transpose of A (no conjugation!) and $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

Write down the Lie algebras and deduce a formula for the dimension of each of the above Lie groups.

3. Show that

(1) $SU(2)$ is diffeomorphic to S^3;
(2) TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$;
(3) if G is a Lie group then TG is diffeomorphic to $G \times \mathbb{R}^d$, where $d = \dim G$;
(4) TS^3 is diffeomorphic to $S^3 \times \mathbb{R}^3$.

In (*ii*)–(*iv*) it is understood that the diffeomorphism should convert the tangent bundle projection into the first projection of the product.

4. Construct an embedding of the n-dimensional torus T^n in $\mathbb{R}^{n+1}$.

5. Determine whether the zero locus $f^{-1}(0)$ of a map $f : \mathbb{R}^n \to \mathbb{R}$ is a $(n-1)$-dimensional (embedded) submanifold of $\mathbb{R}^n$, for each of the following,

(1) $f(x, y, z) = x^2 + y^2 - z^2 + 1$, $(x, y, z) \in \mathbb{R}^3$;
(2) $f(x, y, z) = x^2 + y^2 - z^2$, $(x, y, z) \in \mathbb{R}^3$.
(3) $f(A) = AA^t - I$, where A is a real 3×3 matrix.

6. Prove that the map

$$\rho(x : y : z) = \frac{1}{x^2 + y^2 + z^2}(x^2, y^2, z^2, xy, yz, zx)$$

gives a well-defined *embedding* of $\mathbb{R}P^2$ into $\mathbb{R}^6$. Find on $\mathbb{R}^6$ a finite system of polynomials, of degree ≤ 2, whose common zero locus is precisely the image of ρ. Notice that you needed more than

$4 = \dim \mathbb{R}^6 - \dim \mathbb{R}P^2$ polynomials. (The map ρ is a variant of the 'Veronese embedding', important in Algebraic Geometry.)
Construct an embedding of $\mathbb{R}P^2$ in $\mathbb{R}^4$. [Hint: compose ρ with a suitable map.]

7.* Show that $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$. [Hint: every rotation $A \in SO(3)$ may be written as a composition of two reflections in the planes orthogonal to unit vectors, say $\mathbf{a}$, then $\mathbf{b}$. After $\mathbf{a}$ is chosen, $\mathbf{b}$ is determined up to a ± 1 factor. Express the desired diffeomorphism using the map $(\mathbf{a}, \mathbf{b}) \mapsto (\mathbf{a} \times \mathbf{b}\,,\, \mathbf{a} \cdot \mathbf{b})$ onto $S^3/\pm 1$.]
Deduce that $T(\mathbb{R}P^3)$ is diffeomorphic to $\mathbb{R}P^3 \times \mathbb{R}^3$.

8.* (Whitney's theorem) Rather than using a parameterization, one may want to define curves in $\mathbb{R}^2$ by an equation $F(x, y) = 0$, for a real function F. Show that, however, *every* closed subset of $\mathbb{R}^2$ can be obtained as $F^{-1}(0)$ for some $F \in C^\infty(\mathbb{R}^2)$. [Suggestion: first show that $F^{-1}(0)$ can be the complement of an open disk. The general case uses that $\mathbb{R}^2$ is second countable and requires a careful application of uniform convergence.]

What condition on $F(x, y)$ will eliminate 'unwanted' examples of the zero locus?

9.* (Calabi–Rosenlicht) Let $X = \{(x, y, z) \in \mathbb{R}^3 : xz = 0\}$ be the union of the (x, y)- and (y, z)-coordinate planes and define a family U_a, $a \in \mathbb{R}$, of subsets of X by $U_a = \{(x, y, z) \in X : x \neq 0 \text{ or } y = a\}$ (you might find it helpful to make a sketch of U_a). What is $U_a \cap U_b$ for $a \neq b$?

Now let the map $h_a : U_a \to \mathbb{R}^2$ to the plane $\mathbb{R}^2$ with the coordinates (u_a, v_a) be given by

$$u_a = x, \qquad v_a = \begin{cases} (y - a)/x, & \text{if } x \neq 0 \\ z & \text{if } x = 0 \end{cases}.$$

Show that h_a is a bijection onto $\mathbb{R}^2$, find its inverse, and obtain the formula for $h_b \circ h_a^{-1} : h_a(U_a \cap U_b) \to h_b(U_a \cap U_b)$, for any $a, b \in \mathbb{R}$. Deduce that the family of charts h_a, $a \in \mathbb{R}$, defines a smooth structure on X and that X (with the induced topology from this smooth structure) is Hausdorff, connected, but *not* second countable.

10.

(1) Is $\alpha \wedge \alpha = 0$ true for every differential form α of positive degree?
(2) Let α be a nowhere-zero 1-form. Prove that for a $(p+1)$-form β $(p \geq 0)$, one has $\alpha \wedge \beta = 0$ if and only if $\beta = \alpha \wedge \gamma$ for some p-form γ. [You might like to do it on $\mathbb{R}^n$ first. Partition of unity is useful in the general case.]

11. Prove that $\mathbb{R}P^n$ is orientable if and only if n is odd.
[Hint: consider the $2:1$ map $S^n \to \mathbb{R}P^n$ and a suitable choice of orientation n-form on S^n.]

12. Prove the identity $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$, for a 1-form ω and vector fields X, Y.

13. Show that

$$d\omega = 0, \qquad \text{where} \quad \omega = \frac{-ydx + xdy}{x^2 + y^2},$$

but ω cannot be written as df for any smooth function f on $\mathbb{R}^2 \setminus \{0\}$.
[Hint: consider an appropriate embedding of S^1 in $\mathbb{R}^2$ and integrate the pull-back of ω over S^1.]
Deduce that the de Rham cohomology of the circle is $H^1(S^1) = \mathbb{R}$.

14.

(1) Show that every closed 1-form on S^2 is exact.
(2) Construct a linear isomorphism $H^n(S^n) \cong H^{n-1}(S^{n-1})$, for $n > 1$. Calculate the de Rham cohomology $H^k(S^n)$ for every k, n.
[Suggestion: apply the Poincaré Lemma on the coordinate neighbourhoods for the stereographic projection charts on S^n.]

15. Construct a nowhere-vanishing (smooth) vector field on S^{2n+1} for any n.