

DIFFERENTIAL GEOMETRY EXAMPLES 1

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at g.p.paternain@dpmmms.cam.ac.uk.

1. If X and Y are manifolds, show that $X \times Y$ is a manifold with $\dim X \times Y = \dim X + \dim Y$.
2. Let B_r be the open ball $\{x \in \mathbb{R}^k : |x| < r\}$. Show that the map

$$x \mapsto \frac{rx}{\sqrt{r^2 - |x|^2}}$$

is a diffeomorphism of B_r onto \mathbb{R}^k . (This implies that local parametrizations can always be chosen with all \mathbb{R}^k as domain.)

3. (i) Is the union of two coordinates axis in \mathbb{R}^2 a manifold?
(ii) Prove that the hyperboloid in \mathbb{R}^3 given by $x^2 + y^2 - z^2 = a$ is a manifold for $a > 0$. What happens for $a = 0$? Find the tangent space at the point $(\sqrt{a}, 0, 0)$.
(iii) Show that the solid hyperboloid $x^2 + y^2 - z^2 \leq a$ is a manifold with boundary ($a > 0$).

4. (i) Let $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ be given by $\lambda(x) = e^{-1/x^2}$ for $x > 0$ and $\lambda(x) = 0$ for $x \leq 0$. You know from Analysis I that λ is smooth. Show that $\tau(x) = \lambda(x - a)\lambda(b - x)$ is a smooth function, positive on (a, b) and zero elsewhere ($a < b$).

- (ii) Show that

$$\varphi(x) := \frac{\int_{-\infty}^x \tau}{\int_{-\infty}^{\infty} \tau}$$

is smooth, $\varphi(x) = 0$ for $x < a$, $\varphi(x) = 1$ for $x > b$ and $0 < \varphi(x) < 1$ for $x \in (a, b)$.

- (iii) Finally construct a smooth function on \mathbb{R}^n that equals 1 on the ball of radius a , zero outside the ball of radius b , and is strictly in between at intermediate points (here $0 < a < b$).

These functions are very useful for smooth glueings.

5. Prove that \mathbb{R}^n and \mathbb{R}^m are not diffeomorphic if $n \neq m$.
6. Recall that a submersion is a smooth map $f : X \rightarrow Y$ such that df_x is surjective for all $x \in X$. The *canonical submersion* is the standard projection of \mathbb{R}^k onto \mathbb{R}^l for $k \geq l$, that is

$$(x_1, \dots, x_k) \mapsto (x_1, \dots, x_l).$$

(i) Let f be a submersion, $y = f(x)$. Show that there exist local coordinates around x and y such that f in these coordinates is the canonical submersion.

- (ii) Show that submersions are open maps, i.e. they carry open sets to open sets.

(iii) If X is compact and Y connected, show that every submersion is surjective.

- (iv) Are there submersions of compact manifolds into Euclidean spaces?

7. Show that the orthogonal group $O(n)$ is compact and that its tangent space at the identity is given by all matrices H for which $H^t = -H$.

8. Prove that the set of all 2×2 matrices of rank 1 is a 3-dimensional submanifold of \mathbb{R}^4 .

9. For which values of a does the hyperboloid $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally? What does the intersection look like for different values of a ?

10. Let $f : X \rightarrow X$ be a smooth map. f is called a *Lefschetz map* if given any fixed point x of f , $df_x : T_x X \rightarrow T_x X$ does not have 1 as an eigenvalue. Prove that if X is compact and f is Lefschetz, then f has only finitely many fixed points.

11. (i) Prove that the boundary of manifold with boundary is a manifold without boundary.

(ii) Show that the square $[0, 1] \times [0, 1]$ is not a manifold with boundary.

12. Prove the following theorem due to Frobenius: let A be an $n \times n$ matrix all of whose entries are nonnegative. Then A has a nonnegative real eigenvalue. [Hint: consider the quadrant $\{(x_1, \dots, x_n) \in S^{n-1} : x_i \geq 0 \forall i\}$.]

13. A manifold is said to be *contractible* if the identity map is homotopic to a constant map. Show that a compact manifold without boundary is not contractible.

14. Let X be a compact manifold without boundary and Y a connected manifold with the same dimension as X .

(i) Suppose that $f : X \rightarrow X$ has $\deg_2(f) \neq 0$. Prove that f is onto.

(ii) If Y is not compact, prove that $\deg_2(f) = 0$ for all maps $f : X \rightarrow Y$.

15. Suppose $f : X \rightarrow S^k$ is smooth where X is compact and $0 < \dim X < k$. Let $Z \subset S^k$ be a closed submanifold of dimension complementary to X . Show that $I_2(f, Z) = 0$. Deduce that degrees are the only interesting intersection numbers on spheres.

16 (Morse functions). Let X be a k -manifold and $f : X \rightarrow \mathbb{R}$ a smooth function. Recall that a critical point is a point x for which df_x is not surjective, i.e. $df_x = 0$. A critical point is said to be *non-degenerate* if in local coordinates around x , the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ has non-vanishing determinant. If all the critical points are non-degenerate, f is said to be a *Morse function*.

(i) Show that the condition $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) \neq 0$ is independent of the choice of chart.

(ii) Suppose now that X is an open subset of \mathbb{R}^k . Given $a \in \mathbb{R}^k$, set

$$f_a(x) = f(x) + \langle x, a \rangle,$$

where $\langle x, a \rangle$ denotes the usual inner product in \mathbb{R}^k . Show that f_a is a Morse function for a dense set of values of a . [Hint: consider $\nabla f : X \rightarrow \mathbb{R}^k$.]

With a bit more work one can show that the same result is true if X is now any manifold and not just an open set of Euclidean space. In other words a “generic” smooth function is Morse.

(iii) Show that the determinant function on $M(n)$ is Morse if $n = 2$, but not if $n > 2$.