DYNAMICAL SYSTEMS. EXAMPLES 2.

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`. Most of the examples in this sheet are taken from the text that I am following in lectures: *Introduction to dynamical systems,* by M. Brin & G. Stuck.

1. Let X be a compact metric space and $f : X \to X$ a continuous map. Show that $\omega(x)$ is compact, non-empty and $f(\omega(x)) = \omega(x)$ for every $x \in X$. Give an example showing that $\omega(x)$ may be empty if X is not compact.

2. Show that there are points that are non-recurrent and not eventually periodic for the expanding endomorphism E_m.

3. Give an example of a homeomorphism of a compact metric space which has a dense full orbit but has no dense forward orbits.

4. Let $f : X \to X$ be a homeomorphism. Suppose $h : X \to \mathbb{R}$ is a continuous function such that $h(f^n(x)) = h(x)$ for all $n \in \mathbb{Z}$ and all $x \in X$ (i.e. h is a *first integral*). Show that if f is topologically transitive, h must be constant.

5. Show that the expanding endomorphism E_m is topologically mixing. Prove that (Σ_m^+, σ) is topologically mixing.

6. Show that the expanding endomorphisms E_m, the shifts (Σ_m, σ) and (Σ_m^+, σ), the hyperbolic toral automorphisms, the horseshoe and the solenoid are expansive. Find expansive constants for each.

7. Let $f : X \to X$ be a homeomorphism of a compact metric space and let n be a positive integer. Show that f is expansive if and only if f^n is expansive.

8. Prove that the topological entropy of a map of class C^1 of a compact manifold is finite.

9. Let $g : Y \to Y$ be a factor of $f : X \to X$. Prove that $h(f) \geq h(g)$.

10. Let $f : X \to X$ be a continuous map of a compact metric space. Let $NW_1(f) := NW(f|_{NW(f)})$ and define inductively $NW_i(f) := NW(f|_{NW_{i-1}(f)})$ for $i \geq 2$. Show that a recurrent point belongs to $NW_i(f)$ for all i. Give an example of a map f for which the closure of the set of recurrent points is not equal to $NW(f)$.

11. Calculate the zeta function of a hyperbolic toral automorphism.

12. Find the minimal *positive* value of topological entropy for a topological Markov chain (Σ_A, σ) where A is a 3×3 matrix.

13. Suppose $T : X \to X$ is a continuous transformation of a metric space and μ is a finite T-invariant Borel measure on X whose support is all X. Show that every point is non-wandering and that μ-a.e. point is recurrent.