COLLECTIVE GEODESIC FLOWS

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1. Introduction

Let K be a connected Lie group acting on a Poisson manifold M by Hamiltonian transformations. Let $\psi : M \to \mathfrak{k}^*$ be the associated moment map.

In [9], V. Guillemin and S. Sternberg introduced and studied a class of Hamiltonians that they called *collective*. These functions have the form $f \circ \psi$, where f is a smooth function on $\mathfrak{k}^*$. The Hamiltonian flow ϕ_t of a collective Hamiltonian has a very distinctive feature: its orbits can be reconstructed from the geometry of the action of K and the Hamiltonian flow of f on $\mathfrak{k}^*$. Guillemin and Sternberg introduced collective

Hamiltonians as the mathematical model of various instances in physics in which it is claimed that one physical system moves "as if it were" another physical system. For example, in nuclear physics one has the "liquid drop model" of the nucleus. Besides these physical motivations, collective Hamiltonians form a completely natural class, given a Lie group action.

In the present paper we will focus on what we call *collective geodesic flows.* Suppose that N is a connected manifold on which a connected Lie group K acts. It is well known that the lift of the action of K to T^*N is Hamiltonian with moment map given by

$$\psi(x,p)(\xi) = p(\xi_N(x)), \tag{1}$$

where ξ_N is the vector field on N induced by $\xi \in \mathfrak{k}$ and $p \in T^*_x N$. Suppose that $f : \mathfrak{k}^* \to \mathbb{R}$ is a quadratic form. The special form of ψ ensures that for each $x \in N$, the restriction of $f \circ \psi$ to $T^*_x N$ is also a quadratic form. We call the Hamiltonian flow of such a Hamiltonian a *collective geodesic flow.* Note that in general $f \circ \psi$ restricted to $T^*_x N$ could be degenerate, even if f is not. However, it is quite easy to see using (1) that if the action is *transitive* then $f \circ \psi$ is non-degenerate if f is. Thus, if the action of K is transitive and f is a positive definite quadratic form, then ϕ_t is the geodesic flow of the Riemannian metric determined by $f \circ \psi$. We will see in Section 4 that the set of Riemannian metrics which arise in this fashion is the same as the set of submersion metrics on $N = H\backslash K$ which come from arbitrary left invariant metrics on K. We call them *collective Riemannian metrics.* In general, collective metrics are *not* invariant under the right action of K on $H\backslash K$ because the quadratic Hamiltonians in $\mathfrak{k}^*$ do not need to be Ad^*_K-invariant.

As we explained before, the flow of a collective Hamiltonian can be built from two ingredients: the geometry of the action of K and the Hamiltonian flow of f on $\mathfrak{k}^*$. In the present paper we will prove results that concern these two aspects.

The main dynamical question we would like to address is the following:

When does a collective Hamiltonian have positive topological entropy?

We begin by studying quadratic Hamiltonians on real forms of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$: $so(4) = so(3) \oplus so(3)$, $so(3) \oplus sl(2)$ and $sl(2) \oplus sl(2)$.

Let $\mathfrak{g}$ be a real form of $sl(2;\mathbb{C})$. There exists a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{g}$ such that

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = nX_1, \quad [X_3, X_1] = X_2,$$

where $n = 1$ if $\mathfrak{g}$ is compact, and $n = -1$ otherwise. With respect to this basis of $\mathfrak{g}$ there is the canonical, bi-invariant bilinear form $(,)$ given by

$$(X_i, X_j) = n_{ii}\delta_{ij}$$

where $n_{ii} = n$ if $i = 1$ and 1 otherwise.

For $i = 1, 2$, let $\mathfrak{g}_i$ be a real form of $sl(2;\mathbb{C})$. Let $\langle,\rangle$ denote the bi-invariant inner product on $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\langle,\rangle$ restricted to $\mathfrak{g}_i$ is the inner product $(,)_i$.

Fix $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and let $n_i = 1$ if $\mathfrak{g}_i$ is compact, and -1 otherwise. We identify $\mathfrak{g}^* \simeq \mathfrak{g}$ via the invariant inner product $\langle , \rangle$ and we define for all $x \in \mathfrak{g}_1$, $y \in \mathfrak{g}_2$

$$\begin{aligned} 2H_{I,J}(x,y) &:= (x,Ix)_1 + (y,Jy)_2 = 2H_I(x) + 2H_J(y), \\ &= I_1x_1^2 + I_2x_2^2 + I_3x_3^2 + J_1y_1^2 + J_2y_2^2 + J_3y_3^2, \quad \text{and} \\ H_B(x,y) &:= (x,By)_1, \end{aligned}$$

for diagonal I, J and $B : \mathfrak{g}_2 \to \mathfrak{g}_1$ an arbitrary linear map. The Hamiltonian $H_{I,J}+H_B$ is the general form of a quadratic Hamiltonian on $\mathfrak{g}$.

Denote by C_i the quadratic Casimir on $\mathfrak{g}_i$. In coordinates

$$\begin{aligned} 2C_1(x) &= n_1x_1^2 + x_2^2 + x_3^2, \\ 2C_2(y) &= n_2y_1^2 + y_2^2 + y_3^2, \end{aligned}$$

for $x \in \mathfrak{g}_1$, $y \in \mathfrak{g}_2$. The common level sets of the Casimirs are the closure of the coadjoint orbits.

Define

$$\begin{aligned} a_1 &:= I_3 - I_2, \quad a_2 := n_1I_3 - I_1, \quad a_3 := n_1I_2 - I_1 \\ b_1 &:= J_3 - J_2, \quad b_2 := n_2J_3 - J_1, \quad b_3 := n_2J_2 - J_1. \end{aligned}$$

Note that $n_1a_i, n_2b_i \geq 0$ for all i.

The Casimirs C_1, C_2 are independent integrals of all Hamiltonian vector fields on $\mathfrak{g}$, and for $c_1, c_2 \neq 0$, the common level set $\mathcal{O}_{c_1,c_2} = C_1^{-1}(c_1) \cap C_2^{-1}(c_2)$ is a 4-dimensional symplectic manifold. Because $\{H_I, H_J\}_{\mathfrak{g}} \equiv 0$, it follows that $X_{H_{I,J}}$ is Liouville integrable on each orbit $\mathcal{O}_{c_1,c_2}$ such that $c_1c_2 \neq 0$. We will call these orbits *regular* and the points in them *regular points*.

Our first theorem (proved in Section 2) is based on a careful analysis of a Melnikov-type function. Let $B^{ij} = (X_i, BY_j)_1$:

Theorem A. *Suppose that $\Pi_{i=1}a_ib_i \neq 0$. Let $B : \mathfrak{g}_2 \to \mathfrak{g}_1$ be a non-zero linear map. For all sufficiently small $\epsilon > 0$, the topological entropy of the Hamiltonian flow of $H_\epsilon := H_{I,J} + \epsilon H_B$ is positive on an open set of regular coadjoint orbits of $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$; if $f : \mathfrak{g} \to \mathbb{R}$ is a C^ω first integral of H_ϵ, then f is functionally dependent on H_ϵ, C_1 and C_2. In addition, if $B^{21}B^{23} = 0$, then given a regular coadjoint orbit $\mathcal{O}_{c_1,c_2}$ with c_1 or c_2 positive there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ the topological entropy of the Hamiltonian flow of H_ϵ is positive on $\mathcal{O}_{c_1,c_2}$.*

In Section 2 we explain in detail how this theorem fits in with all the known cases of completely integrable Hamiltonians on $\mathfrak{g}$. These comparisons suggest that the set of completely integrable Hamiltonians (as well as the set of quadratic Hamiltonians with zero entropy) form an algebraic variety whose geometry is yet to be explored. Also as a result of these comparisons we find what appears to be an inconsistency between our results and part 2 of Theorem 3 in [1] (see Section 2).

There is work by A.P. Veselov [25] and by V.V. Kozlov and D.A. Onishchenko [16] which appears related to our Theorem A. Veselov's paper [25] derives non-integrability results for $so(4)$ based on the work by Kozlov and Onishchenko, but his paper only contains statements. Kozlov and Onishchenko study the case of $e(3)$ and their results are based on Melnikov-type calculations. Theorem A cannot be derived from these calculations, however, because they are not carried out in sufficient detail.

Suppose now that $\mathfrak{h}$ is a real semi-simple Lie algebra. If $\mathfrak{h}$ admits an injection of Lie algebras $\mathfrak{g} \hookrightarrow \mathfrak{h}$ then Theorem A plus a simple argument show that $\mathfrak{h}$ admits quadratic Hamiltonians whose flow has positive topological entropy. This naturally raises the question: when does $\mathfrak{h}$ admit an embedding of $\mathfrak{g}$? The answer is given by the following theorem proved in Section 3:

Theorem B. *If $\mathfrak{h}$ is a real semi-simple Lie algebra not isomorphic to a real form of $\mathfrak{a}_1 = sl(2;\mathbb{C})$, $\mathfrak{a}_2 = sl(3;\mathbb{C})$ or to $\mathfrak{a}_1$ or $\mathfrak{a}_2$, then there exists an injection of Lie algebras $\mathfrak{g} \hookrightarrow \mathfrak{h}$, where $\mathfrak{g}$ is a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$, i.e. $\mathfrak{g}$ is a real Lie algebra isomorphic to one of $so(3) \oplus so(3)$, $so(3) \oplus sl(2)$ or $sl(2) \oplus sl(2)$.*

In other words, the theorem says that the only real semi-simple Lie algebras (up to isomorphism) that do not admit an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ are: $sl(2;\mathbb{R}) \simeq su(1,1) \simeq so(2,1)$, $sl(3;\mathbb{R})$, $su(2,1)$, $so(3,1) \simeq sl(2;\mathbb{C})$, $sl(3;\mathbb{C})$, $su(2)$, and $su(3)$.

As we explained before, combining this theorem with Theorem A we can prove:

Corollary. *Let $\mathfrak{h}$ be a real semi-simple Lie algebra not isomorphic to $\mathfrak{a}_i$, $i = 1, 2$, or one of their real forms. Then $\mathfrak{h}$ admits a quadratic Hamiltonian whose flow has positive topological entropy, and contains a subsystem isomorphic to a subshift of finite type.*

It follows from the corollary that most compact semi-simple Lie groups carry many left invariant metrics with positive topological entropy.

We conjecture that the three real forms of the complex simple Lie algebra $\mathfrak{a}_2 = sl(3;\mathbb{C})$ along with $\mathfrak{a}_1 = sl(2;\mathbb{C})$ and $\mathfrak{a}_2 = sl(3;\mathbb{C})$ also admit quadratic Hamiltonians that have chaotic dynamics. The two remaining simple Lie algebras – $sl(2;\mathbb{R})$ and $so(3)$ – obviously do not.

Now that we have addressed our main problem at the level of Lie algebras we turn to the following problem related to the geometry of the moment map of a Hamiltonian action. Let K be a connected Lie group acting by Hamiltonian transformations on a Poisson manifold M. Suppose that $\mathfrak{k}$ admits an embedding of $\mathfrak{g}$ (and we know that most semi-simple Lie algebras do). Let G be a semi-simple Lie group with Lie algebra $\mathfrak{g}$. When does the moment map ψ_G hit a regular orbit? Equivalently we could ask, when does the moment map ψ_K hit an element that projects via $\mathfrak{k}^* \mapsto \mathfrak{g}^*$ to a regular element in $\mathfrak{g}^*$?

In Section 4 we present two criteria that give a positive answer to these questions.

Let $\mathfrak{g}$ be a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ and let G be a semi-simple Lie group with Lie algebra $\mathfrak{g}$. Let M be a Hamiltonian G-space with moment map ψ. We will say that the G action on M is full if there exists an orbit with dimension ≥ 4.

Criterion 1. *Let M be a Hamiltonian G-space, and assume that the G action is full. Then, $\psi(M)$ contains regular coadjoint orbits.*

Suppose that H is a semi-simple Lie group, $\mathfrak{h}$ =Lie(H), and a real form $\mathfrak{g}$ of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ embeds into $\mathfrak{h}$; let $G \leq H$ be a subgroup with Lie algebra $\mathfrak{g}$. Let

$K \subset H$ be a closed subgroup, $\mathfrak{k} \overset{i}{\hookrightarrow} \mathfrak{h}$ its Lie algebra. Because H is semi-simple, we identify $\mathfrak{h}^* \simeq \mathfrak{h}$ via the Cartan-Killing form $(,)$.

Criterion 2. *The moment map $\psi : T^*(K\backslash H) \to \mathfrak{g}^*$ hits a regular orbit iff $\mathfrak{g} \cap \mathfrak{k}^\perp$ contains a regular element.*

Section 4 also contains a formula for the topological entropy of a collective Hamiltonian and an application of the description of collective motion to the Poisson sphere.

Finally in Section 5, using the previous results, we are able to show that many compact homogeneous spaces admit collective Riemannian metrics close to the bi-invariant one with positive topological entropy. These collective metrics Poisson commute with the bi-invariant metric (a remarkable property which comes almost for free).

Theorem C. *For the following homogeneous spaces there are collective Riemannian metrics arbitrarily close to the bi-invariant metric such that they Poisson commute with it and whose geodesic flow has positive topological entropy:*

1. *Spheres: $S^n = SO(n)\backslash SO(n+1)$ for $n \geq 3$;*
2. *Real and complex Stiefel manifolds: $SO(m-r)\backslash SO(m)$ and $SU(m-r)\backslash SU(m)$ for $r \geq 2$ and $m \geq r+2$;*
3. *AI: $SO(m)\backslash SU(m)$ for $m \geq 4$;*
4. *AII: $Sp(m)\backslash SU(2m)$ for $m \geq 2$;*
5. *BDI/AIII: Real oriented and complex Grassmannians: $SO(p)\times SO(q)\backslash SO(p+q)$ and $S(U(p) \times U(q))\backslash SU(p+q)$ for $p = 1, q \geq 3$ and $p, q \geq 2$;*
6. *DIII: $U(m)\backslash SO(2m)$ for $m \geq 2$;*
7. *CI: $U(m)\backslash Sp(m)$ for $m \geq 2$; and*
8. *CII: $Sp(p) \times Sp(q)\backslash Sp(p+q)$ for $p = 1, q \geq 2$ and $p, q \geq 2$.*

The list in the theorem is by no means complete but it is quite illustrative of the techniques developed here which apply to many other spaces. The list does include all simply-connected rank-one symmetric spaces except $\mathbb{C}P^2$, $\mathbb{O}P^2$ (the Cayley projective plane) and the obvious case S^2.

For the geodesic flow, positive topological entropy has a very concrete geometric meaning. Given points p and q in M and $T > 0$, define $n_T(p,q)$ to be the number of geodesic arcs joining p and q with length $\leq T$. R. Mañé [18] showed that

$$h_{top}(g) = \lim_{T\to\infty} \frac{1}{T} \log \int_{M\times M} n_T(p,q)\, dp\, dq,$$

and hence if $h_{top}(g) > 0$, the average number of geodesic arcs between two points grows exponentially. It should be noted that for bi-invariant metrics this number grows polynomially.

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2. Quadratic Hamiltonians on Real Forms of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$

2.1. **Preliminaries.** In this subsection we recall some definitions and terminology and we give some background material on topological entropy.

2.1.1. *Terminology and Definitions.* A *Poisson structure* $\mathcal{P}$ on the smooth manifold M is an element of $\mathrm{Hom}(T^*M, TM)$ such that for any $f \in C^\infty(M)$, the vector field $X_f = \mathcal{P}\, df$ preserves both $\mathcal{P}$ and f. This condition is equivalent to the statement that the bracket $\{f, g\} := \mathcal{P}(df, dg)$ is a Lie bracket on $C^\infty(M)$. A smooth map between Poisson manifolds is a *Poisson map* if it preserves Poisson structures. There are two basic examples of Poisson manifolds: the cotangent bundle of a smooth manifold, and the dual of a Lie algebra. If $\mathfrak{g}$ is a Lie algebra, then we can view $\mathfrak{g} \subset C^\infty(\mathfrak{g}^*)$ by the rule $\xi(\mu) := \langle \mu, \xi \rangle$ for all $\xi \in \mathfrak{g}$, $\mu \in \mathfrak{g}^*$; the Poisson structure on $\mathfrak{g}^*$ is defined for all $\xi, \eta \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$ by $\{\xi, \eta\}(\mu) := -\langle \mu, [\xi, \eta] \rangle$. If $\mathfrak{g}$ is semi-simple, then there is an ad-invariant inner product $(,)$ on $\mathfrak{g}$ that identifies $\mathfrak{g}^* \simeq \mathfrak{g}$. The Poisson bracket on $\mathfrak{g}^*$ pulls-back to a Poisson bracket defined by $[f, g](x) := -(x, [\nabla f_x, \nabla g_x])$ for all $f, g \in C^\infty(\mathfrak{g})$, where ∇ denotes the gradient with respect to $(,)$. The ad-invariance implies that $\dot{x} = X_f(x) = [\nabla f_x, x]$ is the Hamiltonian vector field on $\mathfrak{g}$ associated with this Poisson structure.

A *Casimir* of a Poisson structure $\mathcal{P}$ is an element $f \in C^\infty(M)$ such that $X_f = \mathcal{P}\, df \equiv 0$. Two functions $f, g \in C^\infty(M)$ will be said to be in involution (or to Poisson commute) if $\{f, g\} \equiv 0$. A Casimir Poisson commutes with all smooth functions. The rank of the Poisson structure $\mathcal{P}$ at $x \in M$ is the rank of the linear map $\mathcal{P}_x : T_x^*M \to T_xM$; since $\mathcal{P}$ is skew symmetric the rank is even and $\mathrm{im}\, \mathcal{P}_x$ is a symplectic subspace of T_xM. If we assume that rank $\mathcal{P}_x$ is constant, then $\mathrm{im}\, \mathcal{P}$ generates a distribution that is involutive, and so Frobenius's theorem implies that through each $x \in M$ there is a maximal integral submanifold of this distribtion, M_x, and this distribution comes equipped with a symplectic form. These integral submanifolds are called *symplectic leaves* and locally the foliation of M by symplectic leaves is a fibration, and the components of the fibration map are Casimirs of $\mathcal{P}$ [27]. In the case of the dual of the Lie algebra, the symplectic leaves are the coadjoint orbits and the Casimirs are the invariants of the coadjoint action.

An action of a Lie group G on a Poisson manifold $(M, \mathcal{P})$ is Poisson if there is a Poisson map $\psi : (M, \mathcal{P}) \to (\mathfrak{g}^*, \mathcal{P}_{can})$ such that ψ is G-equivariant. For each $\xi \in \mathfrak{g}$, let ξ_M denote the induced smooth vector field on M; if $\xi_M = X_{\psi^*\xi}$ for all $\xi \in \mathfrak{g}$, then we say the action is *Hamiltonian.* G equivariance implies that for all $\xi \in \mathfrak{g}$, $T\psi(\xi_M) = \mathrm{ad}^*_\xi\, \psi(.)$. Equivalently, if we view $\mathfrak{g} \subset C^\infty(\mathfrak{g}^*)$ as the subspace of linear functions, then because ψ is a Poisson map $\{\psi^*\xi, \psi^*\eta\}(m) = -\langle \psi(m), [\xi, \eta] \rangle = \langle \mathrm{ad}^*_\xi \psi(m), \eta \rangle$. Given a Hamiltonian action of G on $(M, \mathcal{P})$, there is the distinguished subspace $\psi^* C^\infty(\mathfrak{g}^*) \subset C^\infty(M)$. We follow Guillemin and Sternberg and call $H \in \psi^* C^\infty(\mathfrak{g}^*)$ a *collective Hamiltonian.* The map ψ is classically called the momentum or moment map.

2.1.2. *Topological entropy.* In general the topological entropy is defined for an arbitrary continuous flow (or map) on a compact metric space.

Let (X, d) be a compact metric space and let $\phi_t : X \to X$ be a continuous flow. For each $T > 0$ we define a new distance function

$$d_T(x, y) := \max_{0 \le t \le T} d(\phi_t(x), \phi_t(y)).$$

Since X is compact, we can consider the minimal number of balls of radius $\varepsilon > 0$ in the metric d_T that are necessary to cover X. Let us denote this number by $N(\varepsilon, T)$. We define

$$h(\phi, \varepsilon) := \limsup_{T \to \infty} \frac{1}{T} \log N(\varepsilon, T).$$

Observe now that the function $\varepsilon \mapsto h(\phi, \varepsilon)$ is monotone decreasing and therefore the following limit exists:

$$h_{top}(\phi) := \lim_{\varepsilon \to 0} h(\phi, \varepsilon).$$

The number $h_{top}(\phi)$ thus defined is called the *topological entropy* of the flow ϕ_t. Intuitively, this number measures of orbit complexity of the flow. The positivity of $h_{top}(\phi)$ indicates complexity or “chaos” of some kind in the dynamics of ϕ_t. The topological entropy $h_{top}(\phi)$ may also be defined as $h_{top}(\phi_1)$ using the entropy of the time one-map or it may be defined in either of the following ways. All the definitions give the same number $h_{top}(\phi)$ which is independent of the choice of metric [12, 26].

A set $Y \subset X$ is called a (T, ε)-separated set if given different points $y, y' \in Y$ we have $d_T(y, y') \ge \varepsilon$. Let $S(T, \varepsilon)$ denote the maximal cardinality of a (T, ε)-separated set. Then

$$h_{top}(\phi) = \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log S(T, \varepsilon).$$

A set $Z \subset X$ is called a (T, ε)-spanning set if for all $x \in X$ there exists $z \in Z$ such that $d_T(x, z) \le \varepsilon$. Let $M(T, \varepsilon)$ denote the minimal cardinality of a (T, ε)-spanning set. Then

$$h_{top}(\phi) = \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log M(T, \varepsilon).$$

Given a compact subset $K \subset X$ (not necessarily invariant) we can define the topological entropy of the flow with respect to the set K, $h_{top}(\phi, K)$, simply by considering separated (spanning) sets of K.

The following proposition gives an idea of the dynamical significance of the topological entropy (for proofs see [12, 26]).

Proposition 2.1. *The topological entropy verifies the following properties:*

1. *For any two closed subsets Y_1, Y_2 in X,*
$$h_{top}(\phi, Y_1 \cup Y_2) = \max_{i=1,2} h_{top}(\phi, Y_i);$$
2. *If $Y_1 \subset Y_2$ then $h_{top}(\phi, Y_1) \le h_{top}(\phi, Y_2)$;*
3. *Let $\phi_t^i : X_i \to X_i$ for $i = 1, 2$ be two flows and let $\pi : X_1 \to X_2$ be a continuous map commuting with ϕ_t^i i.e. $\phi_t^2 \circ \pi = \pi \circ \phi_t^1$. If π is onto, then $h_{top}(\phi^1) \ge h_{top}(\phi^2)$ and if π is finite-to-one, then $h_{top}(\phi^1) \le h_{top}(\phi^2)$.*

4. *Let $\phi^i_t : X_i \to X_i$ for $i = 1, 2$ be two flows and let $\psi_t := \phi^1_t \times \phi^2_t$ be the product flow on $X_1 \times X_2$. Then $h_{top}(\psi) = h_{top}(\phi^1) + h_{top}(\phi^2)$.*
5. *Given $c \in \mathbb{R}$, let $c\phi_t$ be the flow given by $c\phi_t := \phi_{ct}$. Then $h_{top}(c\phi) = |c|h_{top}(\phi)$.*

2.2. **Sketch of proof of Theorem A.** The proof of Theorem A is based on the Melnikov method. Let us recall this method (we follow Robinson's explanation in [23]): assume $H : M^4 \to \mathbb{R}$ is a smooth function on a compact symplectic manifold, and that F is a second, independent smooth function that Poisson commutes with H. Assume that for a regular value of h, the level set $\{H = h\}$ contains a hyperbolic periodic orbit γ for the flow of ϕ_t of the Hamiltonian vector field X_H. Let $W^{\pm}(\gamma, h)$ denote the stable/unstable manifolds of γ. We assume that $W^+(\gamma, h) = W^-(\gamma, h)$. Let $H_1 \in C^\infty(M^4; \mathbb{R})$ be a second function and $H_\epsilon = H + \epsilon H_1$, ϕ^ϵ_t the flow of H_ϵ. Because γ is hyperbolic, there is a nearby hyperbolic periodic orbit γ_ϵ for ϕ^ϵ_t on $\{H_\epsilon = h\}$, with stable/unstable manifolds $W^{\pm}(\gamma_\epsilon, h)$. In general, $W^+(\gamma_\epsilon, h) \neq W^-(\gamma_\epsilon, h)$ but they do intersect; the points of intersection are called homoclinic points. The Birkhoff-Smale homoclinic theorem tells us that when transverse homoclinic points exist, there is an invariant set in a neighbourhood of γ_ϵ, and ϕ^ϵ_t on this invariant set is equivalent to "flipping a coin." That is, for some cross-section to γ, the first-return map has an invariant set on which it is conjugate to the shift map on the space of bi-infinite sequences of 0s and 1s. These facts imply that the topological entropy of ϕ^t_ϵ is positive.

To detect this dynamical complexity, we must measure the "splitting" of stable and unstable manifolds. To do this, we use the second integral F. By stable manifold theory, γ_ϵ and compact subsets of $W^{\pm}(\gamma_\epsilon, h)$ depend differentiably on ϵ. By stable manifold theory, there is a smooth map $z^{\pm} : W^{\pm}(\gamma, h) \times (-\eta, \eta) \to M$ such that $\operatorname{im} z^{\pm}(., \epsilon) = W^{\pm}(\gamma_\epsilon, h)$ and $z^{\pm}(., 0) = \mathrm{id}$. We can write $G(p, \epsilon) = F \circ z^+(p, \epsilon) - F \circ z^-(p, \epsilon)$; zeros of $G(., \epsilon)$ are points of intersection of stable and unstable manifolds and non-degenerate zeros are transverse points of intersection. We write $G(p, \epsilon) = \epsilon \mathrm{M}(p) + O(\epsilon^2)$, and $\mathrm{M}(p)$ can be computed by:

$$\mathrm{M}(p) = \int_{-\infty}^{\infty} \{F, H_1\} \circ \phi^t(p)\, dt. \tag{2}$$

$\mathrm{M}(p)$ is commonly called the "Melnikov function." By the implicit function theorem, for small ϵ a (non-degenerate) zero of M implies there are (transverse) homoclinic points.

The basic strategy in our proof of Theorem A will be to set up and compute the Melnikov function. A few additional twists appear, though. First, the integrable system ("H") from which we perturb will have two hyperbolic periodic orbits $\gamma_\pm$ such that $W^{+/-}(\gamma_+, h) = W^{-/+}(\gamma_-, h)$. The basic picture remains the same as above, but we must now ensure that the perturbed manifolds do intersect, but do not coincide. Unfortunately, we cannot show this for every hyperbolic periodic orbit. Instead, we must vary the periodic orbits and show that there is at least one pair of periodic orbits where M has the desired properties. This necessitates some involved calculations.

Figure 1 depicts the integrable system and its hyperbolic periodic orbits in schematic fashion.

FIGURE 1. Hyperbolic periodic orbits on a regular coadjoint orbit in $so(4)$.

2.3. **Real Forms of** $sl(2;\mathbb{C})$**.** Let $\mathfrak{g}$ be a real form of $sl(2;\mathbb{C})$, i.e. a real Lie algebra that is isomorphic to either $so(3)$ or $sl(2)$. It is well-known that there exists a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{g}$ such that

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = nX_1, \quad [X_3, X_1] = X_2,$$

where $n = 1$ if $\mathfrak{g} \simeq so(3)$ and $n = -1$ otherwise. With respect to this basis, the canonical bi-invariant bilinear form or Cartan-Killing form $(,)$ is given by

$$(X_i, X_j) = n_{ii}\delta_{ij} \tag{3}$$

where $n_{ii} = n$ if $i = 1$ and 1 otherwise. The automorphism group, $\mathrm{Aut}(\mathfrak{g})$, is also the isometry group of $(,)$. Thus

Lemma 2.2 (Principal Axis Lemma-1)**.** *Let $Q \in S^2(\mathfrak{g}^*)$ be a quadratic form on $\mathfrak{g}$, where $\mathfrak{g}$ is a real form of $sl(2;\mathbb{C})$. Then there exists a $\phi \in \mathrm{Aut}(\mathfrak{g})$ such that for all*

$i, j = 1, 2, 3$

$$(\phi^* Q)(X_i, X_j) = \delta_{ij} I_i \tag{4}$$

for some $I_1, I_2, I_3 \in \mathbb{R}$. If $\mathfrak{g} \simeq so(3)$ then we can assume that $I_1 \leq I_2 \leq I_3$; if $\mathfrak{g} \simeq sl(2)$, then we can assume that $I_3 \leq I_2$.

For $i = 1, 2$, let $\mathfrak{g}_i$ be a real form of $sl(2; \mathbb{C})$. Let $\langle , \rangle$ denote the bi-invariant inner product on $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\langle , \rangle$ restricted to $\mathfrak{g}_i$ is the inner product $(,)_i$. Because $\mathrm{Aut}(\mathfrak{g}_1) \times \mathrm{Aut}(\mathfrak{g}_2) \subset \mathrm{Aut}(\mathfrak{g})$, the following lemma is clear:

Lemma 2.3 (Principal Axis Lemma-2)**.** *Let $Q \in S^2(\mathfrak{g}^*)$ be a quadratic form on $\mathfrak{g}$, where $\mathfrak{g}$ is a real form of $sl(2; \mathbb{C}) \oplus sl(2; \mathbb{C})$. Then there exists $\phi \in \mathrm{Aut}(\mathfrak{g})$ such that*

$$\phi^* Q = \begin{bmatrix} I & B \\ \tilde{B} & J \end{bmatrix} \tag{5}$$

where I and J are in the diagonal form of Lemma 2.2 and $B : \mathfrak{g}_2 \to \mathfrak{g}_1$ is a linear operator, and $\tilde{B} : \mathfrak{g}_1 \to \mathfrak{g}_2$ is its adjoint with respect to the inner products $(,)_i$ on $\mathfrak{g}_i$.

2.4. **Hamiltonians.** Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and let $n_i = 1$ if $\mathfrak{g}_i \simeq so(3)$ and -1 otherwise. We identify $\mathfrak{g}^* \simeq \mathfrak{g}$ via the invariant inner product $\langle , \rangle$ and we define for all $x \in \mathfrak{g}_1$, $y \in \mathfrak{g}_2$

$$\begin{aligned} 2H_{I,J}(x, y) &:= (x, Ix)_1 + (y, Jy)_2 = 2H_I(x) + 2H_J(y), \qquad (6) \\ &= I_1 x_1^2 + I_2 x_2^2 + I_3 x_3^2 + J_1 y_1^2 + J_2 y_2^2 + J_3 y_3^2, \quad \text{and} \\ H_B(x, y) &:= (x, By)_1, \qquad (7) \end{aligned}$$

for diagonal I, J and $B : \mathfrak{g}_2 \to \mathfrak{g}_1$ an arbitrary linear map. On account of the Principal Axis Lemma-2 the Hamiltonian $H_{I,J,B} = H_{I,J} + H_B$ is the general form of a quadratic Hamiltonian on $\mathfrak{g}$.

As in the introduction, denote by C_i the quadratic Casimir on $\mathfrak{g}_i$. In coordinates

$$2C_1(x) = n_1 x_1^2 + x_2^2 + x_3^2, \tag{8}$$

$$2C_2(y) = n_2 y_1^2 + y_2^2 + y_3^2, \tag{9}$$

for $x \in \mathfrak{g}_1$, $y \in \mathfrak{g}_2$.

Define

$$a_1 := I_3 - I_2, \quad a_2 := n_1 I_3 - I_1, \quad a_3 := n_1 I_2 - I_1 \tag{10}$$

$$b_1 := J_3 - J_2, \quad b_2 := n_2 J_3 - J_1, \quad b_3 := n_2 J_2 - J_1. \tag{11}$$

Note that $n_1 a_i, n_2 b_i \geq 0$ for all i. The vector field $X_{H_{I,J}}$ is then given by

$$X_{H_{I,J}}(x, y) = \left\{ \begin{array}{rclrcl} \dot{x}_1 &=& a_1 x_2 x_3, & \dot{y}_1 &=& b_1 y_2 y_3, \\ \dot{x}_2 &=& -a_2 x_1 x_3, & \dot{y}_2 &=& -b_2 y_1 y_3, \\ \dot{x}_3 &=& a_3 x_1 x_2, & \dot{y}_3 &=& b_3 y_1 y_2. \end{array} \right. \tag{12}$$

The Casimirs C_1, C_2 are independent integrals of all Hamiltonian vector fields on $\mathfrak{g}$, and for $c_1, c_2 \neq 0$, the common level set $\mathcal{O}_{c_1,c_2} = C_1^{-1}(c_1) \cap C_2^{-1}(c_2)$ is a 4-dimensional symplectic manifold. Because $\{H_I, H_J\}_{\mathfrak{g}} \equiv 0$, it follows that $X_{H_{I,J}}$ is Liouville integrable on each orbit $\mathcal{O}_{c_1,c_2}$ such that $c_1 c_2 \neq 0$.

For future reference, we compute that

$$
\begin{aligned}
\{H_B, H_I\}_{\mathfrak{g}}(x,y) &= (\dot{x}, By)_1, \\
&= n_1 a_1 x_2 x_3 (B^{11} y_1 + B^{12} y_2 + B^{13} y_3) \\
&\quad - a_2 x_1 x_3 (B^{21} y_1 + B^{22} y_2 + B^{23} y_3) \\
&\quad + a_3 x_1 x_2 (B^{31} y_1 + B^{32} y_2 + B^{33} y_3).
\end{aligned} \tag{13}
$$

2.5. **The orbits.** Henceforth, we assume that $\Pi_{i=1}^3 a_i b_i \neq 0$. An open and dense set of I, J clearly satisfy this condition.

In order to set up the Melnikov integral, we fix the non-zero values for the Casimirs C_1, C_2, and pick orbits of $X_{H_{I,J}}|\mathcal{O}_{c_1,c_2}$ that are heteroclinic connections between two hyperbolic periodic orbits. To do this, we pick a periodic orbit of $X_{H_J}|\mathcal{O}_{c_2}$ and a heteroclinic orbit of $X_{H_I}|\mathcal{O}_{c_1}$ joining the two hyperbolic fixed points $x = \pm\sqrt{2c_1}X_2$ (where $\{X_i\}$ is the canonical $(,)_1$-orthonormal basis of $\mathfrak{g}_1$). Note that $c_1 > 0$ by hypothesis. This is a restriction only if $\mathfrak{g}_1 = sl(2;\mathbb{R})$; in this case, for negative values of c_1 there are only elliptic periodic orbits for X_{H_I} on $\mathcal{O}_{c_1}$.

The formulas for the heteroclinic connections may be found in [14].

2.5.1. *Heteroclinic Connection.* The heteroclinic connections are given by:

$$x_1^{s_1,s_3}(t) = s_1\sqrt{\frac{2c_1a_1}{a_2}}\,\mathrm{sech}(\sqrt{2c_1a_1a_3}t), \tag{14}$$

$$x_2^{s_1,s_3}(t) = -n_1 s_1 s_3\sqrt{2c_1}\tanh(\sqrt{2c_1a_1a_3}t), \tag{15}$$

$$x_3^{s_1,s_3}(t) = s_3\sqrt{\frac{2c_1a_3}{a_2}}\,\mathrm{sech}(\sqrt{2c_1a_1a_3}t), \tag{16}$$

where $s_1, s_3 \in \{\pm 1\}$. The semi-circles in the left half of Figure 1 depicts these connections.

2.5.2. *Periodic Orbits.* The hypothesis on the coefficients J_i implies that $n_2 b_i > 0$ so the ratios $\frac{2n_2h_J - 2J_1c_2}{b_3}$ and $\frac{-2h_J+2J_3c_2}{b_1}$ are positive. Let $\beta_{13}^2 := \frac{2n_2h_J-2J_1c_2}{b_3}$, $\beta_{31}^2 := \frac{-2h_J+2J_3c_2}{b_1}$ and $\gamma := \frac{\beta_{13}}{\beta_{31}}$. For each fixed, non-zero value of c_2, the range of the function $h_J \to \gamma = \gamma(h_J; c_2)$ is $(0,\infty)$. We assume that $0 < \gamma < 1$. Then the periodic orbits of X_{H_J} are given by the following expressions with $r_1, r_3 \in \{\pm 1\}$:

$$y_1^{r_1,r_3}(t) = r_1\,\beta_{31}\sqrt{\frac{b_1}{b_2}}\,\mathrm{dn}_\gamma(\beta_{31}\sqrt{b_1b_3}t), \tag{17}$$

$$y_2^{r_1,r_3}(t) = -r_1r_3\,n_2\beta_{13}\,\mathrm{sn}_\gamma(\beta_{31}\sqrt{b_1b_3}t), \tag{18}$$

$$y_3^{r_1,r_3}(t) = r_3\,\beta_{13}\sqrt{\frac{b_3}{b_2}}\,\mathrm{cn}_\gamma(\beta_{31}\sqrt{b_1b_3}t), \tag{19}$$

where sn_γ, cn_γ and dn_γ are elliptic functions which satisfy $\mathrm{dn}_\gamma' = -\gamma^2\,\mathrm{cn}_\gamma\ \mathrm{sn}_\gamma$, $\mathrm{sn}_\gamma' = \mathrm{cn}_\gamma\ \mathrm{dn}_\gamma$ and $\mathrm{cn}_\gamma' = -\,\mathrm{dn}_\gamma\ \mathrm{sn}_\gamma$ [11].

Although equations (17–19) appear to give 4 distinct periodic orbits of X_{H_J}, in fact only two are actually distinct periodic orbits; for example, the periodic orbits with

$r_1 = r_3 = 1$ and $-r_1 = r_3 = 1$ are distinct, while the remaining two periodic orbits are time translates of these two. See Figure 1.

For future reference, we will denote by $y_j := y_j^{1,1}$. Also note that the solutions for $\gamma > 1$ can be obtained by the following substitutions in equations (17–19): interchange 1 and 3 everywhere they appear, and substitute $\kappa = \gamma^{-1}$ where γ appears.

Let $s = (s_1, s_3)$ and $r = (r_1, r_3)$. The heteroclinic orbits of $X_{H_{I,J}}$ that connect the periodic orbit $\sigma_-^{s,r}(t) = (s_1 s_3 \sqrt{2c_1} X_2, y^r(t))$ and $\sigma_+^{s,r}(t) = (-s_1 s_3 \sqrt{2c_1} X_2, y^r(t))$ are then given by $\sigma^{s,r}(u, \tilde{\theta}) = (x^s(u), y^r(u + \tilde{\theta}))$.

Figure 1 shows the periodic orbits and their stable and unstable manifolds in a schematic fashion. The parameters s, r appear due to the anti-podal $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry.

2.6. **The Melnikov Integral.** The functions $\sigma^{\pm s,r}(u, \tilde{\theta})$ parameterize branches of $W^-(\sigma_-^{s,r}) = W^+(\sigma_+^{s,r})$. The Melnikov integral, which measures the first-order splitting of the unstable manifold of the periodic orbit $\sigma_-^{s,r}(t)$ and the stable manifold of $\sigma_+^{s,r}(t)$ after perturbing the Hamiltonian $H_{I,J}$ by the addition of ϵH_B, is given by (see section 2.2):

$$
\begin{aligned}
M^{s,r}(u, \tilde{\theta}) &:= \int_{-\infty}^{\infty} \{H_B, H_I\} \circ \phi_t(\sigma^{s,r}(u, \tilde{\theta}))\, dt \\
&= \int_{-\infty}^{\infty} \{H_B, H_I\}(x^s(t+u), y^r(t+u+\tilde{\theta}))\, dt \\
&= \sum_{i<j=1,k=1}^{3} \epsilon_{ijk} n_{1i} a_i B^{ik} \int_{-\infty}^{\infty} x_i^{s_1,s_3}(t) x_j^{s_1,s_3}(t) y_k^{r_1,r_3}(t+\tilde{\theta})\, dt
\end{aligned}
\tag{20}
$$

where $n_{1,i} = n_1$ for $i = 1$ and 1 otherwise, ϵ_{ijk} is the sign of the permutation $(i\, j\, k)$ if $\{i, j, k\} = \{1, 2, 3\}$ and 0 otherwise and ϕ_t is the flow of $X_{H_{I,J}}$. $M^{s,r}$ is independent of u because the parameter u is the flow direction.

Let us now explain how to compute $M^{s,r}$: Let $K := \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-\gamma^2 z^2)}}$, $\gamma' = \sqrt{1-\gamma^2}$ and $K' := \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-\gamma'^2 z^2)}}$. K is one-quarter the real period of the elliptic functions sn_γ, cn_γ and dn_γ while $4iK'$ is an imaginary period of each of these meromorphic functions. Let us remark that as $\gamma \to 0$, $K \to \frac{\pi}{2}$, $K' \to \infty$ and $\mathrm{sn}_\gamma \to \sin$, $\mathrm{cn}_\gamma \to \cos$ and $\mathrm{dn}_\gamma \to 1$; as $\gamma \to 1$, $K \to \infty$ and $K' \to \frac{\pi}{2}$ and $\mathrm{sn}_\gamma \to \tanh$, and cn_γ, $\mathrm{dn}_\gamma \to \mathrm{sech}$ [11].

We also let $q = \exp(-\pi \frac{K'}{K})$.

Let

$$
T := \frac{4K}{\beta_{31}\sqrt{b_1 b_3}}, \tag{21}
$$

so that T is the period of the solution $y^r(t)$ to X_{H_J} with $H_J = h_J$ and $C_2 = c_2$. We rescale time by $\tau = \frac{2\pi}{T} t$. We can then write:

$$y_1^r(\tau) = r_1\,\beta_{31}\sqrt{\frac{b_1}{b_2}}\,\mathrm{dn}_\gamma(\frac{2K}{\pi}\tau) = r_1\,\beta_{31}\,\frac{\pi}{2K}\,\sum_{l\in\mathbb{Z}} c_{1,2l}e^{2il\tau}, \tag{22}$$

$$y_2^r(\tau) = -r_1 r_3\,n_2\beta_{13}\,\mathrm{sn}_\gamma(\frac{2K}{\pi}\tau) = r_1 r_3\,\beta_{31}\,\frac{\pi}{2K}\,\sum_{l\in\mathbb{Z}} c_{2,2l+1}e^{(2l+1)i\tau}, \tag{23}$$

$$y_3^r(\tau) = r_3\,\beta_{13}\sqrt{\frac{b_3}{b_2}}\,\mathrm{cn}_\gamma(\frac{2K}{\pi}\tau) = r_3\,\beta_{31}\,\frac{\pi}{2K}\,\sum_{l\in\mathbb{Z}} c_{3,2l+1}e^{(2l+1)i\tau}, \tag{24}$$

where $c_{1,0} = \sqrt{\frac{b_1}{b_2}}$ and for $k \geq 1$, $c_{1,2k} = \sqrt{\frac{b_1}{b_2}}\frac{q^k}{1+q^{2k}} = c_{1,-2k}$; $c_{2,2k+1} = in_2\frac{q^{k+\frac{1}{2}}}{1-q^{2k+1}} = -c_{1,-2k-1}$; and $c_{3,2k+1} = \sqrt{\frac{b_3}{b_2}}\frac{q^{k+\frac{1}{2}}}{1+q^{2k+1}} = c_{3,-2k-1}$ [11]. Since $q = \exp(-\frac{\pi K'}{K})$ the coefficients can also be written as $c_{1,2k} = \sqrt{\frac{b_1}{b_2}}\,\mathrm{sech}(\frac{k\pi K'}{K})$, $c_{2,2k+1} = in_2\,\mathrm{cosech}(\frac{(k+\frac{1}{2})\pi K'}{K})$ and $c_{3,2k+1} = \sqrt{\frac{b_3}{b_2}}\,\mathrm{sech}(\frac{(k+\frac{1}{2})\pi K'}{K})$. The Fourier series (22–24) extend as holomorphic functions on the strip $\{\tau \in \mathbb{C} : |\mathrm{Im}\,\tau| < \frac{\pi K'}{2K}\}$.

For l odd (resp. l even) let $c_{1,l} = 0$ (resp. $c_{2,l} = c_{3,l} = 0$). Let

$$\omega := \frac{T\sqrt{2c_1a_1a_3}}{2\pi}, \tag{25}$$

and we adopt the convention that for $l = 0$, $l\,\mathrm{cosech}(\frac{l\pi}{2\omega}) = \frac{2\omega}{\pi}$. Then, let us define

$$f_j(\theta) := \sum_{l\in\mathbb{Z}} ic_{j,l}l\,\mathrm{sech}(\frac{l\pi}{2\omega})e^{il\theta}, \tag{26}$$

$$g_j(\theta) := \sum_{l\in\mathbb{Z}} c_{j,l}l\,\mathrm{cosech}(\frac{l\pi}{2\omega})e^{il\theta}, \tag{27}$$

for $j = 1, 2, 3$. It is convenient to note that $\gamma = \gamma(h_J, c_2)$ is homogeneous of degree zero in (h_J, c_2), which implies that K, K', q and the coefficients $c_{j,l}$ share this property, while T is homogeneous of degree $-\frac{1}{2}$. The function $\omega = \omega(c_1, h_J, c_2)$ is homogeneous of degree $-\frac{1}{2}$ in (h_J, c_2) and $\frac{1}{2}$ in c_1. The functions $f_j(\theta; c_1, h_J, c_2)$ and $g_j(\theta; c_1, h_J, c_2)$ are therefore homogeneous of degree 0 in (c_1, h_J, c_2).

The functions $c_j : l \to c_{j,l}$ (resp. $t \to x_j^s(t)x_k^s(t)$) decrease exponentially in $|l|$ (resp. $|t|$), so (20) can be calculated by integrating the terms from equations (22-24) term-by-term against $x_j^s(\tau)x_k^s(\tau)$. Using the residue theorem to integrate each term

$x^s_j(\tau)x^s_k(\tau)e^{il\tau}$, we compute that ($\theta = \frac{2\pi}{T}\tilde{\theta}$, $r_2 = r_1 r_3$):

$$\int_{-\infty}^{\infty} x^s_2(t)x^s_3(t)y^r_j(t+\tilde{\theta})\,dt = -n_1\pi\frac{\sqrt{b_1b_3}}{a_1a_3}\left(\frac{\pi\beta_{31}}{2K}\right)^2\sqrt{\frac{a_3}{a_2}}\; s_1r_jf_j(\theta), \tag{28}$$

$$\int_{-\infty}^{\infty} x^s_1(t)x^s_3(t)y^r_j(t+\tilde{\theta})\,dt = \pi\frac{\sqrt{b_1b_3}}{a_1a_3}\left(\frac{\pi\beta_{31}}{2K}\right)^2\sqrt{\frac{a_1a_3}{a_2^2}}\; s_1s_3r_jg_j(\theta), \tag{29}$$

$$\int_{-\infty}^{\infty} x^s_1(t)x^s_2(t)y^r_j(t+\tilde{\theta})\,dt = -n_1\pi\frac{\sqrt{b_1b_3}}{a_1a_3}\left(\frac{\pi\beta_{31}}{2K}\right)^2\sqrt{\frac{a_1}{a_2}}\; s_3r_jf_j(\theta). \tag{30}$$

Let $M^{s,r}(\tilde{\theta}) = -n_1\frac{\pi^3}{4}\sqrt{\left|\frac{b_1b_3}{a_1a_2a_3}\right|}\left(\frac{\beta_{31}}{K}\right)^2 m^{s,r}(\theta)$. Then we get that:

$$\begin{aligned} m^{s,r}(\theta) &= (s_1\sqrt{|a_1|}B^{11} + s_3n_1\sqrt{|a_3|}B^{31})r_1f_1(\theta) \\ &\quad + (s_1\sqrt{|a_1|}B^{12} + s_3n_1\sqrt{|a_3|}B^{32})r_2f_2(\theta) \\ &\quad + (s_1\sqrt{|a_1|}B^{13} + s_3n_1\sqrt{|a_3|}B^{33})r_3f_3(\theta) \\ &\quad + s_1s_3\sqrt{|a_2|}(B^{21}r_1g_1(\theta) + r_1r_3B^{22}g_2(\theta) + B^{23}r_3g_3(\theta)). \end{aligned} \tag{31}$$

Let us remark that $m^{s,r}$ vanishes iff $M^{s,r}$ vanishes, and since the question of when $M^{s,r}$ vanishes will preoccupy us and $m^{s,r}$ is a simpler function, we will utilize the latter and refer to it as the Melnikov function.

2.7. $m^{s_1,s_3}(\theta) \equiv 0$. We will investigate the conditions which imply that $m^{s,r}(\theta)$ changes sign. Because $m^{s,r}(\theta)$ is a real-analytic function of θ, any zero has a finite order of contact and by the results of [23] and [7], the topological entropy of the flow of $H_{I,J} + \epsilon H_B$ is positive for all sufficiently small $\epsilon > 0$. Indeed, there exists a subsystem (or a factor) isomorphic to a Bernoulli subshift of finite type.

For the purposes of this section only, we will assume that $r = (1,1)$, $c_1c_2 \neq 0$ is fixed and we will let $m^s(\theta) = m^{s,r}(\theta; c_1, h_J, c_2)$ for this choice of r and c_1, c_2.

Assume that for some s, $m^s(\theta) \equiv 0$. Taking the mean of $m^s(\theta)$ over $[0, 2\pi]$ and using the fact that only g_1 has a non-zero mean, we compute $0 = s_1s_3\sqrt{|a_2|}\frac{2\omega}{\pi}c_{1,0}B^{21}$ $\Longrightarrow$ $B^{21} = 0$.

Inspection of the coefficients of f_i, g_j shows that g_1, f_2, g_3 are even functions and f_1, g_2, f_3 are odd functions of θ. This means that $m^s(\theta) \equiv 0$ iff:

$$0 \equiv \alpha_1f_1(\theta) + \beta_2g_2(\theta) + \alpha_3f_3(\theta) \tag{32}$$

$$0 \equiv \alpha_2f_2(\theta) + \beta_3g_3(\theta) \tag{33}$$

where α_i (resp. β_j) is the coefficient on f_i (resp. g_j) in equation (31). From the formulas for the $c_{j,l}$, we see that $c_{2,l} = c_{3,l} = 0$ (resp. $c_{1,l} = 0$) for even l (resp. odd l). Equations (26) and (32) imply that $\alpha_1 = 0 \implies s_1\sqrt{|a_1|}B^{11} + s_3n_1\sqrt{|a_3|}B^{31} = 0$.

Equations (32,33) are therefore satisfied iff for all l

$$0 \equiv \beta_2 c_{2,l} + i\alpha_3 c_{3,l} \tanh(\frac{l\pi}{2\omega}) \tag{34}$$

$$0 \equiv i\alpha_2 c_{2,l} \tanh(\frac{l\pi}{2\omega}) + \beta_3 c_{3,l} \tag{35}$$

A necessary condition that there exist a non-trivial solution to (34) is that $\tanh(\frac{l\pi}{2\omega}) = \operatorname{cotanh}(\frac{l\pi K'}{2K})$ for all odd l. Since this is clearly impossible, $\beta_2 = \alpha_3 = 0$. A necessary and sufficient condition that a solution to (35) exist is that $\omega = \frac{K}{K'}$. Since $\lim_{h_J \to n_2 J_1 c_2} \frac{K}{K'} = \lim_{\gamma \to 0} \frac{K}{K'} = 0$ and $\lim_{h_J \to n_2 J_1 c_2} \omega = \lim_{h_J \to n_2 J_1 c_2} \frac{2K}{\pi \beta_{31}} \sqrt{\frac{2c_1 a_1 a_3}{b_1 b_3}} = \frac{b_1}{n_2 b_2 c_2} \sqrt{\frac{2c_1 a_1 a_3}{b_1 b_3}}$, we see that there is no non-trivial solution to (35) on any regular coadjoint orbit $\mathcal{O}_{c_1,c_2}$. Therefore equations (34,35) are satisfied iff $\alpha_i = \beta_j = 0$ for $i, j = 2, 3$. From equation (31) we conclude that $B^{22} = B^{23} = 0$ and that $s_1\sqrt{|a_1|}B^{12} + s_3 n_1 \sqrt{|a_3|}B^{32} = s_1\sqrt{|a_1|}B^{13} + s_3 n_1 \sqrt{|a_3|}B^{33} = 0$. In summary:

Lemma 2.4. *If there is an $s \in \{\pm 1\} \times \{\pm 1\}$ and a regular coadjoint orbit $\mathcal{O}_{c_1,c_2}$ such that $m^s(\theta) \equiv 0$, then $B^{2i} = 0$ and $s_1\sqrt{|a_1|}B^{1i} + s_3 n_1 \sqrt{|a_3|}B^{3i} = 0$ for $i = 1, 2, 3$.*

Because the functions f_j all have zero mean, if $m^s(\theta) \equiv 0$, then all other $m^{s'}$ must have zero mean. Therefore, intersections of other stable and unstable manifolds in the perturbed equations are inevitable, so the 'best' situation that can occur is that $m^{s'}(\theta) \equiv 0$ for all s' – this is a necessary (but not sufficient) condition for the perturbed stable and unstable manifolds to coincide. If $m^{s'}(\theta) \equiv 0$ for all s', then Lemma 2.4 implies that $B = 0$. Thus:

Proposition 2.5. *If there is a regular coadjoint orbit $\mathcal{O}_{c_1,c_2}$ such that for all $s \in \{\pm 1\} \times \{\pm 1\}$, $m^s(\theta) \equiv 0$, then $B^{ij} = 0$ for all $i, j = 1, 2, 3$.*

Lemma 2.4 implies that if $m^s(\theta) \equiv 0$, for some s, then there are two other s' for which $m^{s'}(\theta)$ crosses zero. Thus, at least two of the four heteroclinic connections split, and Burns and Weiss's work [7] implies that the Hamiltonian flow of $X_{H_\epsilon}|\mathcal{O}_{c_1,c_2}$ has positive topological entropy. Since the same argument applies to all $m^{s,r}$ for all r:

Corollary 2.6. *If $H_\epsilon := H_{I,J} + \epsilon H_B$ is a non-trivial perturbation of $H_{I,J}$ such that $X_{H_\epsilon}|\mathcal{O}_{c_1,c_2}$ has zero topological entropy, then $m^{s,r}(\theta; c_1, h_J, c_2)$ is bounded away from 0 for all $s, r \in \{\pm 1\}$.*

We will now show that we cannot bound $m^{s,r}$ away from zero for all s and r and all regular coadjoint orbits $\mathcal{O}_{c_1,c_2}$.

2.8. $m^{s,r}(\theta)$ **is non-zero.** In this section, we investigate some necessary conditions for the Melnikov function $m^{s,r}$ to never have zeros. In this situation, the stable and unstable manifolds of the perturbed hyperbolic periodic orbit no longer intersect; this would be the situation if the perturbed system were also integrable, for example.

Xia [28] gives a sufficient condition for intersections of stable and unstable manifolds of hyperbolic homo- or heteroclinic fixed points to persist under perturbations. Namely, he starts with an exact symplectic manifold (M,ω), $\omega = d\alpha$, and an exact symplectic map $f : (M,\omega) \to (M,\omega)$, $f^*\alpha - \alpha = dS_f$, with hyperbolic fixed points $p_f, q_f \in M$ such that $W^u(p_f) = W^s(q_f)$. If $S_f(q_f) = S_f(p_f)$, then for all sufficiently C^1-close exact symplectic maps g such that $S_g(p_g) = S_g(q_g)$, the stable and unstable manifolds continue to intersect: $W^u(p_g) \cap W^s(q_g) \neq \emptyset$, where p_g, q_g are the perturbed hyperbolic fixed points for g. If this theorem were applicable to the Poincaré map of the flow of $X_{H_\epsilon}|\mathcal{O}_{c_1,c_2}$, then it would clearly preclude the possibility that $m^{s,r}(\theta)$ is nowhere zero. However, a simple argument shows that the theorem is inapplicable in our case.

Let us also observe that $m^{s,r}(\theta; c_1, h_J, c_2)$ is homogeneous of degree 0 in (c_1, h_J, c_2). In the sequel, we will have occasion to compute expressions such as $\lim_{\omega\to 0^+} \frac{2\pi}{\omega} m^{r,s}(\theta;\omega)$. This is shorthand for a limit $\lim_{\lambda\to 0^+} \frac{2\pi}{\omega(\lambda,h_J,c_2)} m^{r,s}(\theta;\lambda,h_J,c_2)$ where h_J, c_2 are fixed positive quantities. We will write $m^{s,r}(\theta;\omega)$ (resp. $m^{s,r}(\theta;\omega,\gamma)$) to emphasize the parametric dependence of $m^{s,r}$ on ω (resp. on both ω and γ).

Let us now sketch the argument why Xia's persistence theorem is inapplicable. Let $\xi := \frac{\pi}{2\omega}$. As $\xi \to +\infty$, we have that $\xi \sinh(l\xi) \to 0$, and $\xi\,\mathrm{cosech}(l\xi) \to 0$ for all $l \neq 0$. This implies that $\frac{\pi}{2\omega} g_j(\theta;\omega) \to c_{j,0}$ and $\frac{\pi}{2\omega} f_j(\theta;\omega) \to 0$. Recall that $c_{1,0}$ is a function only of H_J and C_2, so these limits can be taken by letting $c_1 \to 0$ while holding H_J and C_2 fixed. If $\gamma < 1$ (resp. $\gamma > 1$), then only $c_{1,0} \neq 0$ (resp. $c_{3,0} \neq 0$), so

$$\lim_{\omega\to 0^+} \frac{\pi}{2\omega} m^{s,r}(\theta;\omega) = s_1 s_3 r_1 \sqrt{|a_2|} B^{2j} c_{j,0}, \tag{36}$$

where $j = 1$ (resp. $j = 3$). With $B^{21} \neq 0$ (resp. $B^{23} \neq 0$), this will be non-zero, and so for small values of ω we see that $m^{s,r}(\theta;\omega)$ will be bounded away from zero. This implies that the intersection of the stable and unstable manifolds of the two hyperbolic fixed points (of the Poincaré map) $\pm\sqrt{2c_2}X_2$ does not persist on an open set of regular coadjoint orbits. A more careful argument shows that on each regular coadjoint orbit there are hyperbolic periodic orbits for which $m^{s,r}$ is bounded away from zero.

Let us now investigate necessary conditions for $m^{s,r}(\theta;\omega,\gamma)$ to be non-zero for all values of ω, γ. We will prove that for a fixed s, $m^{s,r}(\theta;\omega,\gamma)$ can be non-zero for all $\gamma, \omega > 0$ for at most 2 values of r. This will imply that for the remaining values of r, $m^{s,r}(\theta;\omega,\gamma)$ must alternate sign. This will prove Theorem A.

Let $\xi := \frac{\pi}{2\omega}$ as above. As $\xi \to 0^+$, we have that $\xi^{-1}\sinh(\xi) \to 1$, $\sinh\xi\,\mathrm{cosech}(l\xi) \to \frac{1}{l}$, and $\sinh\xi\,\mathrm{sech}(l\xi) \to 0$. This implies that $\sinh(\frac{\pi}{2\omega}) g_j(\theta;\omega) \to \frac{2K}{\pi\beta_{31}} y_j(\theta)$ and $\sinh(\frac{\pi}{2\omega}) f_j(\theta;\omega) \to 0$ as $\omega \to \infty$ (see (22–24)). Then:

$$\lim_{\omega\to\infty} s_1 s_3 \sinh(\frac{\pi}{2\omega}) m^{s,r}(\theta;\omega) = \frac{2K}{\pi\beta_{31}} \sqrt{|a_2|}(B^{21}y_1^r(\theta) + B^{22}y_2^r(\theta) + B^{32}y_3^r(\theta)), \tag{37}$$

Recall that the functions y_j^r are parametrized by the constants of motion h_J and c_2, so the limit in (37) can be taken by letting $c_1 \to \infty$ while holding h_J and c_2 fixed.

Let us assume that for some s and r, $s_1 s_3 m^{s,r}(\theta;\omega) > 0$ for all $\omega > 0$ and $\theta \in \mathbb{R}$; since $\sinh(\frac{\pi}{2\omega}) > 0$ for all $\omega > 0$, the expression in (37) is non-negative. Using equations (17–19) and multiplying the limit in (37) by $\frac{\pi}{2\gamma K}\sqrt{\frac{|b_2|}{|a_2|}}$ we arrive at:

$$B^{21}\sqrt{|b_1|}\; r_1\gamma^{-1}\operatorname{dn}_\gamma(\theta) - B^{22}\sqrt{|b_2|}\; n_2 r_1 r_3 \operatorname{sn}_\gamma(\theta) + B^{23}\sqrt{|b_3|}\; r_3 \operatorname{cn}_\gamma(\theta) \geq 0, \tag{38}$$

$\forall\gamma < 1$ and $\forall\theta$. Taking the limit of the LHS in (38) as $\gamma \to 1$ then gives

$$B^{21}\sqrt{|b_1|}\; r_1 \operatorname{sech}(\theta) - B^{22}\sqrt{|b_2|}\; n_2 r_1 r_3 \tanh(\theta) + B^{23}\sqrt{|b_3|}\; r_3 \operatorname{sech}(\theta) \geq 0 \tag{39}$$

$\forall\theta$. Letting $\theta \to \pm\infty$ shows that $B^{22} = 0$.

Inequality (38) now implies that $B^{21}\sqrt{|b_1|}\; r_1\gamma^{-1}\operatorname{dn}_\gamma(\theta) + B^{23}\sqrt{|b_3|}\; r_3 \operatorname{cn}_\gamma(\theta) \geq 0$ for all θ and $0 < \gamma < 1$. Since $\operatorname{dn}_\gamma(\theta+K) = \operatorname{dn}_\gamma(\theta)$, $\operatorname{cn}_\gamma(\theta+K) = -\operatorname{cn}_\gamma(\theta)$, we also have that $B^{21}\sqrt{|b_1|}\; r_1\gamma^{-1}\operatorname{dn}_\gamma(\theta) - B^{23}\sqrt{|b_3|}\; r_3 \operatorname{cn}_\gamma(\theta) \geq 0$. Since $\gamma^{-1}\operatorname{dn}_\gamma(0) = \operatorname{cn}_\gamma(0) = 1$, we conclude that

$$B^{21}\sqrt{|b_1|}\; r_1 \pm B^{23}\sqrt{|b_3|}\; r_3 \geq 0. \tag{40}$$

In summary,

Lemma 2.7. *If for some s and r the Melnikov function $m^{s,r}$ is nowhere zero for all $\omega > 0$ and all $0 < \gamma < 1$, then $B^{22} = 0$ and*

$$M\left(B^{21}\sqrt{|b_1|}\; r_1 \pm B^{23}\sqrt{|b_3|}\; r_3\right) \geq 0, \tag{41}$$

where $M = \operatorname{sign} m^{s,r}$.

Let us return to the case where, for a given r, $s_1 s_3 m^{s,r}(\theta;\omega,\gamma) \geq 0$ for all $\omega > 0$ and $0 < \gamma < 1$. Let us examine the case where $\kappa = \gamma^{-1} < 1$. In this case, the formula for $m^{s,r}(\theta;\omega,\gamma)$ can be obtained from $m^{s,r}(\theta;\omega,\kappa)$ by interchanging the subscripts 1 and 3 everywhere. Equation (37) becomes

$$s_1 s_3 \frac{\pi}{2\kappa K}\sqrt{\frac{|b_2|}{|a_2|}} \lim_{\omega\to\infty} \sinh(\frac{\pi}{2\omega}) m^{s,r}(\theta;\omega,\kappa) = B^{23}\sqrt{|b_3|}\; r_3\kappa^{-1}\operatorname{dn}_\kappa(\theta) + B^{21}\sqrt{|b_1|}\; r_1 \operatorname{cn}_\kappa(\theta). \tag{42}$$

Taking the limit as $\kappa \to 1$ shows that $s_1 s_3 m^{s,r}(\theta;\omega,\gamma) \geq 0$ for all $\omega > 0$ and $\gamma > 1$, also. Arguments similar to those above now prove that

$$B^{23}\sqrt{|b_3|}\; r_3 \pm B^{21}\sqrt{|b_1|}\; r_1 \geq 0. \tag{43}$$

Since the case where $s_1 s_3 m^{s,r}(\theta;\omega,\gamma) \leq 0$ for $0 < \gamma < 1$ is similar, we can combine inequalities (40, 43) and conclude that:

Lemma 2.8. *If for some s and r, the Melnikov function $\theta \to m^{s,r}(\theta;\omega,\gamma)$ is nowhere zero for all values of $\omega > 0$ and $\gamma > 0$, then $B^{21}\sqrt{|b_1|}\; r_1 = B^{23}\sqrt{|b_3|}\; r_3$ and $B^{22} = 0$.*

Proposition 2.9. *There does not exist a Melnikov function $m^{s,r}$ such that for all $s, r \in \{\pm 1\} \times \{\pm 1\}$ and all $\omega, \gamma > 0$, the function $\theta \to m^{s,r}(\theta;\omega,\gamma)$ is nowhere zero.*

Proof: If such an $m^{s,r}$ existed, then because $\Pi_{i=1}^{3} a_i b_i \neq 0$, lemma (2.8) implies that $B^{2i} = 0$ for $i = 1, 2, 3$. Therefore, for all s and r and ω, γ, $m^{s,r}(\theta; \omega, \gamma)$ is a zero-mean function (equation 31). Absurd. □

Theorem A. *Suppose that $I \in S^2(\mathfrak{g}_1^*)$ and $J \in S^2(\mathfrak{g}_2^*)$ satisfy $\Pi_{i=1} a_i b_i \neq 0$. Let $B : \mathfrak{g}_2 \to \mathfrak{g}_1$ be a non-zero linear map. For all sufficiently small $\epsilon > 0$, the topological entropy of the Hamiltonian flow of $H_\epsilon := H_{I,J} + \epsilon H_B$ on an open set of regular coadjoint orbits of $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is positive; if $f : \mathfrak{g} \to \mathbb{R}$ is a C^ω first integral of H_ϵ, then f is functionally dependent on H_ϵ, C_1 and C_2. If $B^{21}B^{23} = 0$ then given a regular coadjoint orbit $\mathcal{O}_{c_1,c_2}$ with c_1 or c_2 positive there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ the topological entropy of the Hamiltonian flow of H_ϵ is positive on $\mathcal{O}_{c_1,c_2}$.*

Proof. Let $\phi_t^\epsilon : \mathfrak{g} \to \mathfrak{g}$ denote the flow of X_{H_ϵ} and let $c_1 c_2 \neq 0$. Suppose that $h_{top}(\phi_1^\epsilon | \mathcal{O}_{c_1,c_2}) = 0$, for all $c_1 c_2 \neq 0$. Then, for each pair (r, s) and all $\omega, \gamma > 0$ $m^{s,r}(\theta; \omega, \gamma)$ is either nowhere zero or it is identically zero. This is impossible by Propositions 2.4 and 2.9. Therefore, there is an (r, s) and $\gamma, \omega > 0$ such that $m^{s,r}(\theta; \omega, \gamma)$ crosses zero. By continuity, there is an open set of regular coadjoint orbits such that $m^{s,r}(\theta; c_1, h_J, c_2)$ crosses zero.

If $B^{21} = 0$ (resp. $B^{23} = 0$), then for $\gamma < 1$ (resp. $\gamma > 1$) and all $\omega > 0$ $m^{r,s}(\theta; \omega, \gamma)$ is a zero-mean, non-identically zero function.

□

2.9. **Remarks.**

Remark 2.10. Let us consider several well-known families of integrable geodesic flows on $so(4)$. Let $\Lambda : so(4) \to so(4)$ be a symmetric, non-degenerate linear map that is diagonal in the standard basis of $so(4)$, with distinct eigenvalues Λ_{ij}. If there exist diagonal matrices α, β with the eigenvalues of α pairwise distinct such that $\forall X \in so(4)$

$$[X, \beta] + [\alpha, \Lambda X] \equiv 0, \tag{44}$$

then $\Lambda_{ij} = \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j}$. The Hamiltonian $2h(X) = \sum_{i<j} \Lambda_{ij} X_{ij}^2$ has an additional family of independent first integrals $\sum_{i<j} \frac{\gamma_i - \gamma_j}{\alpha_i - \alpha_j} X_{ij}^2$ for arbitrary $\gamma_i \in \mathbb{R}$ [2, 1, 10]. The Mischenko-Manakov [19, 17] case of the rigid body is of this form with $\beta = \alpha^2$.

Let $\mathcal{R} \subset C^\omega(so(4))$ denote the set of $h = h_\Lambda$ described in the previous paragraph. We will now show that for each $H_{I,J}$ satisfying the hypothesis of Theorem A, there is an open neighbourhood of $H_{I,J}$ in the complement of $\mathcal{R}$ – since $\mathcal{R}$ is closed, it suffices to establish that all such $H_{I,J} \in \mathcal{R}^c$.

Let $X \in so(4)$ be given by

$$X = \begin{bmatrix} 0 & u_1 & -u_2 & v_3 \\ -u_1 & 0 & u_3 & v_2 \\ u_2 & -u_3 & 0 & v_1 \\ -v_3 & -v_2 & -v_1 & 0 \end{bmatrix}, \tag{45}$$

and define the map $\phi : X \to x \oplus y \in so(3) \oplus so(3)$ by

(46)

$$\phi(X) = \begin{bmatrix} 0 & u_1 + v_1 & -u_2 + v_2 \\ -u_1 - v_1 & 0 & u_3 + v_3 \\ u_2 - v_2 & -u_3 - v_3 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & u_1 - v_1 & -u_2 - v_2 \\ -u_1 + v_1 & 0 & u_3 - v_3 \\ u_2 + v_2 & -u_3 + v_3 & 0 \end{bmatrix},$$

which is a Lie algebra isomorphism. The Hamiltonian $2h(x \oplus y) = \sum_{i=1}^3 I_i x_i^2 + J_i y_i^2$ is transformed to $2h \circ \phi(X) = \sum_{i=1}^3 (I_i + J_i)(u_i^2 + v_i)^2 + 2(-1)^{i+1}(I_i - J_i)u_i v_i$. For $I \neq J$, $h \circ \phi$ is non-diagonal. For $I = J$, a simple argument shows that if

$$\Lambda_{12} = \Lambda_{34} = 2I_1, \tag{47}$$

$$\Lambda_{13} = \Lambda_{24} = 2I_2, \tag{48}$$

$$\Lambda_{23} = \Lambda_{14} = 2I_3, \quad \text{and} \tag{49}$$

$$\Lambda_{ij} = \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j} \quad \forall i, j, \tag{50}$$

then $a_1 a_2 a_3 = 0$, i.e. the coefficients I_i are not all distinct.

Indeed, if there is a solution to these four equations, then the set of α_i, β_j satisfying the four equations is 3-dimensional, due to the invariance under the 3-dimensional group of transformations $\alpha_i \to c\alpha_i + a$, $\beta_j \to c\beta_j + b$. This implies that the rank of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & I_1 & -I_1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & I_1 & -I_1 \\ 1 & 0 & -1 & 0 & I_2 & 0 & -I_2 & 0 \\ 0 & 1 & 0 & -1 & 0 & I_2 & 0 & -I_2 \\ 0 & 1 & -1 & 0 & 0 & I_3 & -I_3 & 0 \\ 1 & 0 & 0 & -1 & I_3 & 0 & 0 & -I_3 \end{bmatrix} \tag{51}$$

is less than or equal to 5. The determinant of the 6×6 matrix obtained by striking columns 1 and 5 from A is $2(I_3 - I_2)(I_3 - I_1)(I_2 - I_1) = -2a_1 a_2 a_3$. This shows that $H_{I,J} \in \mathcal{R}$ implies that $I = J$ and $a_1 a_2 a_3 = 0$. Thus, if $H_{I,J}$ satisfies the hypothesis that $\Pi_{i=1}^3 a_i b_i \neq 0$ of Theorem A, then $H_{I,J} \in \mathcal{R}^c$.

Remark 2.11. One construction of Mishchenko-Fomenko [20] of integrable quadratic Hamiltonians on real semi-simple Lie algebras $\mathfrak{g}$ can be described quite succinctly. Let $\mathfrak{t}$ be a maximal abelian subalgebra, and let $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ be an $\mathrm{ad}_{\mathfrak{t}}$-invariant decomposition of $\mathfrak{g}$. For $t \in \mathfrak{t}$ in general position, $\mathrm{ad}_t : \mathfrak{p} \to \mathfrak{p}$ is a linear isomorphism. We let $a, b \in \mathfrak{t}$ be two elements in general position and let $\phi : \mathfrak{t} \to \mathfrak{t}$ be a non-degenerate, symmetric linear map. Then, we define for all $x = t + p \in \mathfrak{t} + \mathfrak{p}$

$$H_{\phi,a,b}(x) := \langle \phi(t), t \rangle + \langle \mathrm{ad}_b^{-1} \circ \mathrm{ad}_a p, p \rangle, \tag{52}$$

where $\langle , \rangle$ is the Cartan-Killing form on $\mathfrak{g}$. Mischenko-Fomenko [21] show that these Hamiltonians are integrable on all semi-simple $\mathfrak{g}$. Indeed, their proof shows that $H_{\phi,a,b}$ is $\mathrm{ad}_{\mathfrak{t}}$-invariant, which implies that $H_{\phi,a,b}$ has rank $\mathfrak{g}$ linear first integrals.

Let us consider the case when $\mathfrak{g} = so(3) \oplus so(3)$ is a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$. In this case, rank $\mathfrak{g} = 2$ and all maximal abelian subalgebras are conjugate to the standard $\mathfrak{t} = so(2) \oplus so(2)$. $H_{\phi,a,b}$ is $so(2) \oplus so(2)$ invariant, and relative to the decomposition $\mathfrak{g} = so(3) \oplus so(3)$, we have that

$$2H_{\phi,a,b}(x \oplus y) = \phi_{11}x_1^2 + 2\phi_{12}x_1y_1 + \phi_{22}y_1^2 + \alpha(x_2^2 + x_3^2) + \beta(x_2^2 + x_3^2), \tag{53}$$

where $a = \alpha_1 X_1 + \alpha_2 Y_1$, $b = \beta_1 X_1 + \beta_2 Y_1$, $\alpha = \frac{\alpha_1}{\beta_1}$, $\beta = \frac{\alpha_2}{\beta_2}$ and $\alpha_i, \beta_j, \phi_{ij} \in \mathbb{R}$. It is clear that $H_{I,J}$ is of this form iff $I_2 = J_2$, $I_3 = J_3$ and $\Pi_{i=1}^3 a_i b_i = 0$.

The remaining cases are similar.

Remark 2.12. In [1], Adler and van Moerbeke claim that on $\mathfrak{g} = so(4) = so(3) \oplus so(3)$, there exist integrable Hamiltonians of the form

$$2H = \sum_{i=1}^{6} \lambda_i z_i^2 + \sum_{i=1}^{3} \lambda_{i,i+3} z_i z_{i+3}, \tag{54}$$

for certain choices of the coefficients $\lambda_{i,i+3}$, depending on the λ_j. In the notation of the current paper, $z = (x, y)$, $\lambda_i = I_i$, $\lambda_{3+i} = J_i$, and $\lambda_{i,i+3} = B^{ii}$ for $i = 1, 2, 3$. Their conditions are that

$$(\lambda_{14}^2, \lambda_{25}^2, \lambda_{36}^2)F^2 = \left[\frac{(A_{25} - A_{36})^2}{A_{32}A_{65}}, \frac{(A_{36} - A_{14})^2}{A_{13}A_{46}}, \frac{(A_{14} - A_{25})^2}{A_{21}A_{54}}\right] E, \tag{55}$$

and

$$E\lambda_{14}\lambda_{25}\lambda_{36} \neq 0, \tag{56}$$

where $A_{ij} := \lambda_i - \lambda_j$, $E := A_{32}A_{65}A_{13}A_{46}A_{21}A_{54}$ and $F := A_{46}A_{32} - A_{65}A_{13}$.

If $\lambda_i = a + ci$, then $A_{ij} = c(i - j)$ and one readily computes that $F = 0$, $E \neq 0$, and conditions (55,56) are satisfied for all values of $\lambda_{i,i+3}$ such that $\lambda_{14}\lambda_{25}\lambda_{36} \neq 0$.

In the notation of the current paper, $\Pi_{i=1}^3\ a_i b_i = 4c^6$, so Theorem A implies that there does not exist an additional, independent real-analytic first integral of the Hamiltonian (54) for $\lambda_i = a + ci$ and all $\lambda_{i,i+3}$ sufficiently small – which contradicts Theorem 3, part 2 of [1]. We assume that part 2 requires the additional hypothesis that $F \neq 0$, but we are unable to see why.

Remark 2.13. In [4], Bogoyavlenskij constructs a number of examples of integrable quadratic Hamiltonians on $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{g}_i \simeq so(3)$ or $sl(2)$. The Hamiltonians he considers, in the notation employed here, are of the form

$$2H(x, y) = \sum_{i=1}^{3} I_i x_i^2 + 2B^{ii} x_i y_i + J_i y_i^2. \tag{57}$$

Bogoyavlenskij shows that for each pair (I, J) with the property that $\Pi_{i=1}^{3} a_i b_i \neq 0$, there exists a non-zero choice of the coefficients B^{ii} such that the Hamiltonian H is integrable with a second quadratic integral.

3. Embeddings of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ into semi-simple Lie algebras

We will prove the following theorem:

Theorem B. *If $\mathfrak{h}$ is a real semi-simple Lie algebra not isomorphic to a real form of $\mathfrak{a}_1 = sl(2;\mathbb{C})$, $\mathfrak{a}_2 = sl(3;\mathbb{C})$ or to $\mathfrak{a}_1$ or $\mathfrak{a}_2$, then there exists an injection of Lie algebras $\mathfrak{g} \hookrightarrow \mathfrak{h}$, where $\mathfrak{g}$ is a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$, i.e. $\mathfrak{g}$ is a real Lie algebra ismorphic to one of $so(3) \oplus so(3)$, $so(3) \oplus sl(2)$ or $sl(2) \oplus sl(2)$.*

Assuming this theorem, we prove:

Corollary. *If $\mathfrak{h}$ is a real semi-simple Lie algebra not isomorphic to a real form of $\mathfrak{a}_1 = sl(2;\mathbb{C})$, $\mathfrak{a}_2 = sl(3;\mathbb{C})$ or to $\mathfrak{a}_1$ or $\mathfrak{a}_2$, then $\mathfrak{h}$ admits a non-degenerate quadratic Hamiltonian whose flow has positive topological entropy, and contains a subsystem isomorphic to a subshift of finite type.*

Proof. Let $\mathfrak{h}$ be a real semi-simple Lie algebra not isomorphic to one of the named Lie algebras. Then there exists a real form $\mathfrak{g}$ of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ that injects into $\mathfrak{h}$. Let $i : \mathfrak{g} \hookrightarrow \mathfrak{h}$ denote this injection. Let $(,)$ denote the Cartan-Killing form on $\mathfrak{h}$, and $((,))$ denote the induced ad-invariant bilinear form on $\mathfrak{g}$. Because $\mathfrak{g}$ is semi-simple, $((,))$ is ad-invariant, and i is injective, $((,))$ is non-degenerate. This implies that $(,)$ is non-degenerate on $i(\mathfrak{g})^{\perp} \subset \mathfrak{h}$, too. Let $\mathfrak{h} = i(\mathfrak{g}) + i(\mathfrak{g})^{\perp}$ be the $(,)$-orthogonal decompostion. We write $x \in \mathfrak{h}$ as the unique sum $x = y + z$ for $y \in i(\mathfrak{g})$ and $z \in i(\mathfrak{g})^{\perp}$. Then we define:

$$H(x) := H_{\epsilon}(y) + F(z), \tag{58}$$

where $H_{\epsilon}(y) = H_{I,J}(y) + \epsilon H_B(y)$ in the notation of the previous section and $F(z) = \frac{1}{2}(z, z)$. Because $(,)$ is non-degenerate on $i(\mathfrak{g})^{\perp}$, for all B and all ϵ sufficiently small, H is a non-degenerate quadratic form on $\mathfrak{h}$. The associated Hamiltonian vector field is

$$X_H(y + z) = [\nabla_y H_{\epsilon}, y + z] + [z, y], \tag{59}$$

for all $x \in \mathfrak{h}$. In the case that $z = 0$, we have $X_H(y) = [\nabla_y H_{\epsilon}, y]$, which since $i(\mathfrak{g})$ is a subalgebra and $y, \nabla_y H_{\epsilon} \in i(\mathfrak{g})$ for all $x = y + z \in \mathfrak{h}$, we see that $i(\mathfrak{g})$ is invariant under X_H and $X_H|i(\mathfrak{g}) = Ti(X_{H_{\epsilon}})$. We now apply the work of the previous section. □

3.1. Rank ≥ 3 case.

Let $\mathfrak{h}$ be a real, semi-simple Lie algebra, $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. Recall that a real form of a complex Lie algebra $\mathfrak{c}$ is a real Lie algebra $\mathfrak{h}$ such that $\mathfrak{h}^{\mathbb{C}}$ is isomorphic to $\mathfrak{c}$ as a complex Lie algebra. A real form of $sl(2;\mathbb{C})$ is a real Lie algebra isomorphic to either $su(2) \simeq so(3)$ or $sl(2)$. In the following, $\mathfrak{g}$ will denote a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$, so $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_i \simeq so(3)$ or $sl(2)$.

Let $\mathfrak{c}$ be a semi-simple, complex Lie algebra. Let $\mathfrak{c} = \mathfrak{t} + \sum_{\alpha\in\Delta} \mathfrak{c}^\alpha$ be the root-space decomposition of $\mathfrak{c}$ relative to a Cartan subalgebra $\mathfrak{t}$. Here t_α is the unique element in $\mathfrak{t}$ such that for all $t \in \mathfrak{t}$, $\alpha(t) = (t_\alpha, t)$, where $(,)$ is the Cartan-Killing form. Since $(,)$ is non-degenerate on $\mathfrak{t}$, it induces a non-degenerate bilinear form $\langle , \rangle$ on $\mathfrak{t}^*$. The element x^α is a basis of $\mathfrak{c}^\alpha$ such that $[x^\alpha, x^{-\alpha}] = 2t_\alpha$ and $\mathrm{ad}_t x^\alpha = \alpha(t)x^\alpha$ for each $\alpha \in \Delta$. The set $\Delta \subset \mathfrak{t}^*$ is the set of *roots* of $\mathfrak{c}$. A set $B \subset \Delta$ is a *basis* of Δ if it spans $\mathfrak{t}^*$ and for each $\alpha \in \Delta$, there exist unique integers $n_{\alpha,\beta}$, all of the same sign, such that $\alpha = \sum_{\beta\in B} n_{\alpha,\beta}\beta$. The matrix $A = [A_{\alpha,\beta}]$ with $A_{\alpha,\beta} := 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ for all $\alpha, \beta \in B$ is called the Cartan matrix [13]. Finally, the *rank* of $\mathfrak{c}$ is the dimension of $\mathfrak{t}$.

Let us introduce the following notion: we will say that a real semi-simple Lie algebra $\mathfrak{h}$ has *compact rank* r if there exists an r-dimensional, maximal compact abelian subalgebra $\mathfrak{s} \subset \mathfrak{h}$. If the compact rank of $\mathfrak{h}$ equals the rank of $\mathfrak{h}^{\mathbb{C}}$, then $\mathfrak{s}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{h}^{\mathbb{C}}$. Conversely, if $\mathfrak{s} \subset \mathfrak{h}$ is a compact abelian subalgebra of $\mathfrak{h}$ and $\mathfrak{s}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{h}^{\mathbb{C}}$, then the compact rank of $\mathfrak{h}$ equals the rank of $\mathfrak{h}^{\mathbb{C}}$.

Lemma 3.1. *Let $\mathfrak{h}$ be a semi-simple, real Lie algebra of compact rank ≥ 2; suppose that the compact rank of $\mathfrak{h}$ equals the rank of $\mathfrak{h}^{\mathbb{C}}$. If there exist roots $\alpha, \beta \in \Delta(\mathfrak{h}^{\mathbb{C}})$ such that $\alpha \pm \beta \notin \Delta(\mathfrak{h}^{\mathbb{C}})$, then there exists a real form $\mathfrak{g}$ of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ and an embedding $i : \mathfrak{g} \hookrightarrow \mathfrak{h}$.*

Let us remark that if $\alpha, \beta \in \Delta$ and $\alpha \pm \beta \notin \Delta$, then $\langle\alpha, \beta\rangle = 0$.

Proof. Let $\mathfrak{s} \subset \mathfrak{h}$ be a maximal abelian compact subalgebra such that $\mathfrak{s}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{h}^{\mathbb{C}}$. Let $t_\alpha, t_\beta \in \mathfrak{s}^{\mathbb{C}}$ denote the elements such that $\sigma = (t_\sigma, .)$ for $\sigma = \alpha, \beta$. Since $\mathfrak{s}$ is a compact subalgebra of $\mathfrak{h}$, it is clear that $it_\alpha, it_\beta \in \mathfrak{s}$. Since $\mathfrak{s}^{\mathbb{C}}$ is a Cartan subalgebra, there exists a unique subspace $\mathfrak{h}^{\alpha,\mathbb{C}}$ (resp. $\mathfrak{h}^{\beta,\mathbb{C}}$) of 2 complex dimensions in $\mathfrak{h}^{\mathbb{C}}$ that is ad_s-invariant for all $s \in \mathfrak{s}^{\mathbb{C}}$ and ad_s has eigenvalues $\pm\alpha(s)$ on $\mathfrak{h}^{\alpha,\mathbb{C}}$ (resp. $\pm\beta(s)$ on $\mathfrak{h}^{\beta,\mathbb{C}}$). Let $\mathfrak{h}^\sigma := \mathfrak{h}^{\sigma,\mathbb{C}} \cap \mathfrak{h}$ for $\sigma = \alpha, \beta$. For each $s \in \mathfrak{s}$, ad_s is a real transformation that has eigenvalues $\pm\sigma(s)$ on $\mathfrak{h}^{\sigma,\mathbb{C}}$; hence, ad_s leaves $\mathfrak{h}^\sigma$ invariant for $\sigma = \alpha, \beta$. In particular, the subspaces $\mathfrak{h}^\alpha$, and $\mathfrak{h}^\beta$ are both invariant under it_α, and it_β. By hypothesis, $0 = \langle\alpha, \beta\rangle = \alpha(t_\beta) = \beta(t_\alpha)$, so the action of it_α on $\mathfrak{h}^\beta$ is trivial and vice versa. Since $\alpha \pm \beta \notin \Delta$, $[\mathfrak{h}^{\alpha,\mathbb{C}}, \mathfrak{h}^{\beta,\mathbb{C}}] = 0$ and so $[\mathfrak{h}^\alpha, \mathfrak{h}^\beta] = 0$. Finally, since $[\mathfrak{h}^{\sigma,\mathbb{C}}, \mathfrak{h}^{\sigma,\mathbb{C}}] = \mathbb{C}t_\sigma$ and $\mathfrak{h}^\sigma$ is real, it follows that $[\mathfrak{h}^\sigma, \mathfrak{h}^\sigma] = \mathbb{C}t_\sigma \cap \mathfrak{h} = \mathbb{R}it_\sigma$ for $\sigma = \alpha, \beta$.

Therefore, $\mathfrak{g} := \mathbb{R}it_\alpha + \mathfrak{h}^\alpha + \mathbb{R}it_\beta + \mathfrak{h}^\beta \subset \mathfrak{h}$ is a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$. □

Corollary 3.2. *Let $\mathfrak{h}$ be a real, semi-simple but not simple Lie algebra. Then there is an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ into $\mathfrak{h}$.*

Proof. Since $\mathfrak{h}$ is not simple, $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ where each $\mathfrak{h}_i$ is semi-simple and non-trivial. Because the Cartan-Killing form is not positive-definite on $\mathfrak{h}_i$, each $\mathfrak{h}_i$ has compact rank ≥ 1. In addition, $\Delta(\mathfrak{h}^{\mathbb{C}}) = \Delta(\mathfrak{h}_1^{\mathbb{C}}) \oplus \Delta(\mathfrak{h}_2^{\mathbb{C}})$ is an $\langle , \rangle$ direct sum with the property that if $\alpha \in \Delta(\mathfrak{h}_1^{\mathbb{C}})$, $\beta \in \Delta(\mathfrak{h}_2^{\mathbb{C}})$ then $\alpha \pm \beta \notin \Delta(\mathfrak{h}^{\mathbb{C}})$. □

Corollary 3.3. *Let $\mathfrak{u}$ be a compact real form of one of $\mathfrak{a}_l$, $\mathfrak{b}_l$, $\mathfrak{c}_l$, $\mathfrak{d}_l$ for $l \geq 3$ or $\mathfrak{b}_2$, $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$. Then $\mathfrak{u}$ admits an embedding of $so(3) \oplus so(3)$.*

Proof. Let $\mathfrak{u}$ denote a compact real form of one of the listed Lie algebras – with the exception of $\mathfrak{b}_2$ and $\mathfrak{g}_2$. The compact rank of $\mathfrak{u}$ equals the rank of $\mathfrak{u}^{\mathbb{C}}$. The rank of each of the algebras in question is l which is ≥ 3. An examination of the Cartan matrix of each of these Lie algebras reveals that there are $\alpha, \beta \in B$ such that $A_{\alpha,\beta} = 0$, which implies that $\langle \alpha, \beta \rangle = 0$. Since $\alpha - \beta \notin \Delta$ by the definition of B, their orthogonality implies that $\alpha \pm \beta \notin \Delta$. Hence, $\mathfrak{u}$ satisfies the conditions of Lemma 3.1, and since the real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ must be compact, this lemma follows for the considered algebras.

If $\mathfrak{u}$ is the compact real form of $\mathfrak{b}_2 = so(5;\mathbb{C})$, then $\mathfrak{u} = so(5)$ admits the obvious embedding of $so(3) \oplus so(3) = so(4)$. (One can also employ Lemma 3.1 here, but this is overkill).

Finally, let $\mathfrak{u}$ be the compact real form of $\mathfrak{g}_2$. The root system of $\mathfrak{g}_2$ is $\Delta(\mathfrak{g}_2) = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\beta+2\alpha), \pm(\beta+3\alpha), \pm(2\beta+3\alpha)\}$ and the roots $\alpha, 2\beta+3\alpha$ are $\langle , \rangle$-orthogonal [13]. Hence, we can apply Lemma 3.1 to obtain the injection. ☐

We note that up to isomorphism the only complex, simple Lie algebras missing from the list in the previous lemma are $\mathfrak{a}_1 = sl(2;\mathbb{C})$ and $\mathfrak{a}_2 = sl(3;\mathbb{C})$. Clearly, a compact real form of $\mathfrak{a}_1$ ($= so(3)$) cannot admit an embedding of $so(3) \oplus so(3)$. We will now establish that there is no embedding of any real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ into a compact real form of $\mathfrak{a}_2$.

3.2. **Rank 2 Case.**

Lemma 3.4. *Let $\mathfrak{h}$ be a semi-simple, real Lie algebra of rank 2. If there exists an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$, then there exist non-zero roots $\alpha, \beta \in \Delta(\mathfrak{h}^{\mathbb{C}})$ that are $\langle , \rangle$-orthogonal.*

Proof. Let $\mathfrak{g}$ denote the real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ that injects into $\mathfrak{h}$, and let i be the injection. Then $i^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \hookrightarrow \mathfrak{h}^{\mathbb{C}}$ extends to an injection of the complexified Lie algebras. Let $\mathfrak{t}_0$ denote the standard Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}} = sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ and $\mathfrak{t}_1 = i^{\mathbb{C}}(\mathfrak{t}_0)$. Because the rank of $\mathfrak{h}$ is two, $\mathfrak{t}_1$ is a Cartan subalgebra of $\mathfrak{h}^{\mathbb{C}}$.

By the injectiveness of $i^{\mathbb{C}}$, there exists a 6-dimensional subalgebra of $\mathfrak{h}^{\mathbb{C}}$ with basis $h_\lambda, h_\mu, x^{\pm\lambda}, x^{\pm\mu}$ such that $[x^\sigma, x^{-\sigma}] = 2h_\sigma$ for $\sigma = \lambda, \mu$, and $[h_\lambda, h_\mu] = 0$, $[x^\lambda, x^{\pm\mu}] = [x^{-\lambda}, x^{\pm\mu}] = 0$. The Cartan subalgebra $\mathfrak{t}_1 = \text{span}\{h_\lambda, h_\mu\}$ since the rank of $\mathfrak{h}^{\mathbb{C}}$ is two. In addition, there exist linear functionals $\alpha, \beta \in \mathfrak{t}_1^*$ such that $[h, x^{\pm\lambda}] = \pm\alpha(h)x^{\pm\lambda}$ (resp. $[h, x^{\pm\mu}] = \pm\beta(h)x^{\pm\mu}$) for all $h \in \mathfrak{t}_1$. Thus, $\pm\alpha, \pm\beta \in \Delta$ are roots of $\mathfrak{h}^{\mathbb{C}}$ and $x^{\pm\lambda}$ (resp. $x^{\pm\mu}$) spans $\mathfrak{h}^{\pm\alpha,\mathbb{C}}$ (resp. $\mathfrak{h}^{\pm\beta,\mathbb{C}}$). Therefore, $[\mathfrak{h}^{\alpha,\mathbb{C}}, \mathfrak{h}^{\pm\beta,\mathbb{C}}] = 0$ and so $\alpha \pm \beta \notin \Delta$ which implies $\langle \alpha, \beta \rangle = 0$. ☐

Corollary 3.5. *Let $\mathfrak{h}$ be a real, simple Lie algebra with compact rank 2. Then $\mathfrak{h}$ admits an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ iff $\Delta(\mathfrak{h}^{\mathbb{C}})$ possesses two non-zero $\langle , \rangle$ orthogonal roots.*

In particular, no real form of $\mathfrak{a}_2 = sl(3;\mathbb{C})$ admits an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$.

A real form $\mathfrak{h}$ of the exceptional Lie algebra $\mathfrak{g}_2$ admits an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ iff there is a maximal compact subalgebra $\mathfrak{k} \subset \mathfrak{h}$ of rank 2. In particular, the compact real form of $\mathfrak{g}_2$ admits an embedding of $so(3) \oplus so(3)$.

Proof. The necessity of the condition on the compact rank of $\mathfrak{h}$ and $\Delta(\mathfrak{h}^{\mathbb{C}})$ follows from Lemma 3.4. Their sufficiency derives from inspection of the root systems of rank 2 simple Lie algebras, along with Lemma 3.1. Since the root system $\Delta(\mathfrak{a}_2) = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$, the orthogonality condition on the roots of $\Delta(\mathfrak{a}_2)$ cannot be satisfied for any real form of $\mathfrak{a}_2$. The remaining claim follows from Lemma 3.1. □

3.3. **Noncompact Case.** Let $\mathfrak{h}$ be a real, semi-simple Lie algebra such that $\mathfrak{h}$ does not admit an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$. By Corollary 3.2, $\mathfrak{h}$ must be simple. It is known that a real simple Lie algebra is either a complex simple Lie algebra viewed as a real Lie algebra, or it is a real form of a complex simple Lie algebra. We now examine a these alternatives for $\mathfrak{h}$.

Case 1: If $\mathfrak{h}$ is a complex simple Lie algebra, viewed as a real Lie algebra, then the compact real form of $\mathfrak{h}$ is a real subalgebra of $\mathfrak{h}$. By Lemma 3.3, only the compact real forms of $\mathfrak{a}_1 = sl(2;\mathbb{C})$ and $\mathfrak{a}_2 = sl(3;\mathbb{C})$ do not admit a real embedding of a real from of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$. Hence $\mathfrak{h} \simeq \mathfrak{a}_1$ or $\mathfrak{a}_2$.

Case 2-i: If $\mathfrak{h}$ is a real form of a complex simple Lie algebra, then either it is compact or not. If $\mathfrak{h}$ is compact, then Corollary 3.3 implies that $\mathfrak{h}$ is a compact real form of $\mathfrak{a}_1$ or $\mathfrak{a}_2$.

Case 2-ii: If $\mathfrak{h}$ is a real non-compact, simple Lie algebra, then let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{h}$ and let $\mathfrak{h} = \mathfrak{k} + \mathfrak{p}$ denote the $(,)$-orthogonal Cartan decomposition of $\mathfrak{h}$ with respect to $\mathfrak{k}$. Since $\mathfrak{k}$ admits an ad-invariant, non-degenerate bilinear form, it is reductive so $\mathfrak{k} = \mathfrak{a} \oplus \mathfrak{u}$ where $\mathfrak{a}$ is abelian and $\mathfrak{u}$ is semi-simple. It is possible that either $\mathfrak{a} = \{0\}$ or $\mathfrak{u} = \{0\}$ but both cannot simultaneously be trivial. Since $\mathfrak{h}$ does not admit an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$, neither does $\mathfrak{u}$, so Corollary 3.2 implies that $\mathfrak{u}$ is simple. Lemma 3.3 therefore implies that $\mathfrak{u}$ is a compact real form of $\mathfrak{a}_1$ or $\mathfrak{a}_2$ if it is non-trivial. Thus, $\mathfrak{u} \simeq \{0\}$, $su(2)$, or $su(3)$.

Table V, p. 518 in [13] provides a list of all complex simple Lie algebras, their real forms, and the maximal compact subalgebras of these real forms. By inspection, the only non-compact simple real Lie algebras with maximal compact subalgebra isomorphic to one of $\mathfrak{a}$, $\mathfrak{a} \oplus su(2)$ or $\mathfrak{a} \oplus su(3)$ for some abelian algebra $\mathfrak{a}$ are: (AI) $\mathfrak{h} \simeq sl(2;\mathbb{R}), sl(3;\mathbb{R})$; (AIII) $\mathfrak{h} \simeq su(1,1)$, $su(2,1)$, $su(3,1)$, $su(2,2)$, $su(3,2)$; (BDI) $\mathfrak{h} \simeq so(2,1)$, $so(3,1)$, $so(2,2)$, $so(3,2)$; (DIII) $\mathfrak{h} \simeq so^*(4)$, $so^*(6)$; (CI) $\mathfrak{h} = sp(1;\mathbb{R})$, $sp(2;\mathbb{R})$, $sp(3;\mathbb{R})$ where $sp(n;\mathbb{R})$ is the Lie algebra of symplectic linear transformations of $\mathbb{R}^{2n}$. This exhausts the list of real, simple non-compact Lie algebras that may not admit an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$.

We will now eliminate several of these candidates. By the special isomorphism $so(2,2) \simeq sl(2) \oplus sl(2)$ ([13], p. 520) we see that $so(2,2)$, $so(3,2) \simeq sp(2;\mathbb{R})$, $su(2,2)$, $sp(3;\mathbb{R})$ and $su(3,2)$ all admit an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ (again, Lemma 3.1 can be applied directly, but this is not needed). In addition, $so^*(4) \simeq su(2) \oplus sl(2)$. For $su(3,1)$, we have $\mathfrak{k} = s(u(3) \oplus u(1))$; therefore the compact rank of $su(3,1)$ is at least 3. Because $su(3,1)$ is a real form of $\mathfrak{a}_3 = sl(4;\mathbb{C})$, we see that the compact rank of $su(3,1)$ equals the rank of $su(3,1)^{\mathbb{C}}$. In addition, by inspection of the Cartan matrix of $\mathfrak{a}_3$ there exist roots $\alpha, \beta \in \Delta(\mathfrak{a}_3)$ such that

$\alpha \pm \beta \notin \Delta(\mathfrak{a}_3)$. By Lemma 3.1, $su(3,1)$ admits an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$. Finally, we have $so^*(6) \simeq su(3,1)$.

To summarize case 2-ii: The non-compact, real forms of complex simple Lie algebras (up to isomorphism) that may not admit an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ are: $sl(2;\mathbb{R}) \simeq su(1,1) \simeq so(2,1)$, $sl(3;\mathbb{R})$, $su(2,1)$, $so(3,1) \simeq sl(2;\mathbb{C})$. Each of the listed Lie algebras has compact rank ≤ 1, so none admit an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$.

Therefore, combining Cases 1 and 2, the only real semi-simple Lie algebras (up to isomorphism) that do not admit an embedding of a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ are: $sl(2;\mathbb{R}) \simeq su(1,1) \simeq so(2,1)$, $sl(3;\mathbb{R})$, $su(2,1)$, $so(3,1) \simeq sl(2;\mathbb{C})$, $sl(3;\mathbb{C})$, $su(2)$, and $su(3)$. That is, we have proven Theorem B.

4. Collective motion

Let K be a connected Lie group acting on a Poisson manifold M by Hamiltonian transformations. Let $\psi : M \to \mathfrak{k}^*$ denote the associated moment map. Recall that a Hamiltonian H on M is said to be *collective* if it is a pullback by the moment map of a smooth function f on $\mathfrak{k}^*$, i.e. $H = f \circ \psi$.

Let us explain what ingredients go into the solution of a collective Hamiltonian. The discussion below is taken from [8].

Recall that a function $f \in C^\infty(\mathfrak{k}^*)$ defines a map $L_f : \mathfrak{k}^* \to \mathfrak{k}$ by the formula: $L_f(c)(\alpha) = df_c(\alpha)$. The map L_f is sometimes known as the *Legendre transformation* associated with f. The following important relation holds:

$$X_H(x) = L_f(\psi(x))_M(x) \quad \text{for all } x \in M. \tag{60}$$

It follows from equation (60) that if $x(t)$ denotes the trajectory of the Hamiltonian system X_H with $x(0) = x$, then $x(t)$ lies entirely on the orbit of K through x and hence $\psi(x(t))$ lies entirely on the orbit $\mathcal{O}$ through $\psi(x)$. Moreover $\gamma(t) = \psi(x(t))$ is a solution of the Hamiltonian system corresponding to $f_\mathcal{O} := f|_\mathcal{O}$. Set $\xi(t) := L_f(\gamma(t))$, then equation (60) says that $\dot{x}(t) = \xi(t)_M(x(t))$. So we can find the solution curve by applying the following three steps:

(1) Find the orbit $\mathcal{O}$ through $\psi(x)$.

(2) Find the solution to the Hamiltonian system on $\mathcal{O}$ corresponding to $f_\mathcal{O}$ passing through $\psi(x)$ at $t = 0$. Call this curve $\gamma(t)$.

(3) Compute the curve $\xi(t) = L_f(\gamma(t))$. This is a curve in $\mathfrak{k}$. Solve the differential equation (i.e. find the curve in K satisfying)

$$\dot{a}(t) = \xi(t)_M a(t), \quad a(0) = e.$$

Then $a(t)x$ is the desired solution curve.

Remark 4.1. Suppose for instance that f is invariant. Then, on each orbit $\mathcal{O}$, $\gamma(t)$ is constant, but the map L_f need not be trivial. Thus $\xi(t)$ will be a constant element of $\mathfrak{k}$, and $a(t)$ will be a one-parameter subgroup. Therefore the motion corresponding to $f \circ \psi$ when f is invariant is given by the action of a one-parameter subgroup, the one-parameter subgroup depending on x.

4.1. **Entropy formula for collective Hamiltonians.** Let K be a compact Lie group . Let M be a Poisson manifold on which K acts by Hamiltonian transformations with moment map $\psi : M \to \mathfrak{k}^*$. Take a collective Hamiltonian $H = f \circ \psi$ and let ϕ_t denote the flow of X_H. In what follows, for a subset $A \subset M$, we will denote $h_{top}(\phi, A)$ also by $h_{top}(H, A)$.

Recall that $\mathcal{O}_c$ denotes the orbit through c under the coadjoint action, and $f_{\mathcal{O}_c}$ stands for the restriction of f to $\mathcal{O}_c$.

Proposition 4.2. *If $A \subset M$ is any compact ϕ_t-invariant subset we have*

$$h_{top}(H, A) = \sup_{c \in \psi(A)} h_{top}(f_{\mathcal{O}_c}, \mathcal{O}_c \cap \psi(A)).$$

Proof. As we mentioned in the last section, ϕ_t leaves the orbits of K invariant. Hence it follows from Corollary 18 in [6] that

$$h_{top}(H, A) = \sup_{x \in A} h_{top}(H, A \cap \mathcal{O}_x), \tag{61}$$

where $\mathcal{O}_x$ denotes the orbit of K through x. Let us compute now $h_{top}(H, A \cap \mathcal{O}_x)$.

Consider the map $\pi := \psi|_{\mathcal{O}_x} : \mathcal{O}_x \to \mathcal{O}_{\psi(x)}$. By Theorem 17 in [6] we deduce that

$$h_{top}(H, \mathcal{O}_x) \leq h_{top}(f_{\mathcal{O}_{\psi(x)}}, \mathcal{O}_{\psi(x)}) + \sup_{c \in \mathcal{O}_{\psi(x)}} h_{top}(H, \pi^{-1}(c)).$$

But now, according to the description of collective motion that we gave above, the curve $a(t)$ is the same for every $x \in \psi^{-1}(c)$. Hence for any $x \in \psi^{-1}(c)$, $\phi_t x = a(t)x$ and since K is compact this clearly implies $h_{top}(H, \pi^{-1}(c)) = 0$. Thus

$$h_{top}(H, \mathcal{O}_x) \leq h_{top}(f_{\mathcal{O}_{\psi(x)}}, \mathcal{O}_{\psi(x)}).$$

But note that the reverse inequality also holds since ψ takes orbits of H to orbits of f. Thus

$$h_{top}(H, \mathcal{O}_x) = h_{top}(f_{\mathcal{O}_{\psi(x)}}, \mathcal{O}_{\psi(x)}),$$

and also

$$h_{top}(H, \mathcal{O}_x \cap A) = h_{top}(f_{\mathcal{O}_{\psi(x)}}, \mathcal{O}_{\psi(x)} \cap \psi(A)).$$

This equation together with equation (61) implies

$$h_{top}(H, A) = \sup_{x \in A} h_{top}(f_{\mathcal{O}_{\psi(x)}}, \mathcal{O}_{\psi(x)} \cap \psi(A)) = \sup_{c \in \psi(A)} h_{top}(f_{\mathcal{O}_c}, \mathcal{O}_c \cap \psi(A)).$$

□

Corollary 4.3. *Suppose the energy level $H^{-1}(a) = \psi^{-1}(f^{-1}(a))$ is compact. Then*

$$h_{top}(H, H^{-1}(a)) = \sup_{c \in f^{-1}(a)} h_{top}(f_{\mathcal{O}_c}, f_{\mathcal{O}_c}^{-1}(a)).$$

Let us discuss some applications.

Example 4.4. Let X be a compact Hamiltonian $SO(3)$-space. In this case the coadjoint orbits are two-spheres. Hence for any smooth function on $so(3)^*$ we have $h_{top}(f_{\mathcal{O}_c}) = 0$. Thus we deduce that for any collective Hamiltonian H, $h_{top}(H) = 0$.

Example 4.5. Let K be a compact Lie group endowed with a left invariant metric. Then it is known that its associated Hamiltonian is collective for the right action [8, pag 219]. Let f denote the quadratic form on $\mathfrak{k}^*$ that defines the left invariant metric. Then, the corollary implies that the topological entropy of the geodesic flow defined by a left invariant metric g is given by

$$h_{top}(g) = \sup_{c \in f^{-1}(1)} h_{top}(f_{\mathcal{O}_c}, f_{\mathcal{O}_c}^{-1}(1)).$$

We deduce for example, that for $K = SO(3)$, $h_{top}(g) = 0$. However, the results in Section 3 show that for most compact semi-simple Lie groups there will be left invariant metrics with positive topological entropy.

Example 4.6. Let X be a compact Hamiltonian G-space with moment map $\psi : X \to \mathfrak{g}^*$. Let $K \subset G$ be a closed subgroup. The inclusion $\mathfrak{k} \to \mathfrak{g}$ induces a projection $\pi : \mathfrak{g}^* \to \mathfrak{k}^*$. This projection, restricted to a coadjoint orbit $\mathcal{O}$, can be viewed as the moment map corresponding to Hamiltonian action of K on $\mathcal{O}$. Now let $f : \mathfrak{k}^* \to \mathbb{R}$ be a function invariant under the coadjoint action of K on $\mathfrak{k}^*$. Set $H = f \circ \pi \circ \psi$. Apply now the proposition twice; once to deduce that $h_{top}(f \circ \pi_{\mathcal{O}_c}, \mathcal{O}_c) = 0$ and again to obtain $h_{top}(H) = 0$.

Collective functions like H, i.e., Hamiltonians defined by means of a subalgebra and the corresponding projection, were introduced by Thimm [24] to prove the complete integrability of certain geodesic flows on homogeneous spaces.

4.2. **Submersions and collective metrics.** Let M be a Riemannian manifold on which the group H acts freely, properly and by isometries. Consider the quotient $B = H \backslash M$ and let $\rho : M \to B$ be the canonical projection. Endow B with the submersion metric. The metrics on M and B induce canonical maps $TM \xrightarrow{\chi_1} T^*M$ and $TB \xrightarrow{\chi_2} T^*B$. Suppose now that K is a group acting on M and its action commutes with the action of H. Then there is a naturally induced action on B. In this way, by lifting to the corresponding cotangent bundles, we have two moment maps: $\psi_K^1 : T^*M \to \mathfrak{k}^*$ and $\psi_K^2 : T^*B \to \mathfrak{k}^*$.

Proposition 4.7. *The equality $\psi_K^2 \circ \chi_2 \circ d\rho = \psi_K^1 \circ \chi_1$ holds on the set of horizontal vectors in TM.*

Proof. Recall that the moment map ψ_K^1 is given by $\psi_K^1(m,p)(\zeta) = p(\zeta_M(m))$, where ζ_M is the vector field on M induced by $\zeta \in \mathfrak{k}$. Similarly $\phi_K^2(b,u)(\zeta) = u(\zeta_B(b))$. Hence we need to prove that if $v \in T_mM$ is horizontal, then

$$\langle d_m\rho(v), \zeta_B(\rho(m))\rangle \;=\; \langle v, \zeta_M(m)\rangle.$$

But observe that $\zeta_B(\rho(m)) = d_m\rho(\zeta_M(m))$. Hence

$$\langle d_m\rho(v), \zeta_B(\rho(m))\rangle \;=\; \langle d_m\rho(v), d_m\rho(\zeta_M(m))\rangle.$$

But since v is horizontal by the definition of the submersion metric

$$\langle d_m\rho(v), d_m\rho(\zeta_M(m))\rangle \;=\; \langle v, \zeta_M(m)\rangle$$

as desired.

□

We apply the proposition to the following situation. Let K be a Lie group with a left-invariant metric. Then $TK \stackrel{\chi_1}{\to} T^*K$ gives rise to a left-invariant Hamiltonian on T^*K. Then it is known that the latter is collective for the right action of K on T^*K [8, pag 219]. In other words our Hamiltonian can be written as $f \circ \psi_K^1$ where f is some positive definite quadratic form on $\mathfrak{k}^*$. Now let H be a subgroup of K acting from the left. Then we can endow $H\backslash K$ with the submersion metric. Clearly there is also an induced action of K on $T^*(H\backslash K)$ with moment map ψ_K^2. Then from the proposition we deduce that $f \circ \psi_K^2$ is the Hamiltonian associated with the submersion metric on $H\backslash K$. We have proved:

Corollary 4.8. *The Hamiltonian associated with the submersion metric on $H\backslash K$ is collective for the canonical action of K on $T^*(H\backslash K)$, and its defining function is the same one that defines the left-invariant metric on K.*

Remark 4.9. Collective metrics in general are *not* invariant under the right action of K on $H\backslash K$ because the quadratic Hamiltonians in $\mathfrak{k}$ do not need to be Ad_K-invariant.

4.3. **The Poisson sphere.** In this subsection we will study collective Riemannian metrics on S^2. All these metrics are completely integrable and their geodesic flows have zero topological entropy.

Let H_I be the Hamiltonian on $so(3)$ given by:

$$2H_I(x) = \frac{x_1^2}{I_1} + \frac{x_2^2}{I_2} + \frac{x_3^2}{I_3}.$$

We will assume in what follows that $I_3 > I_2 > I_1 > 0$.

The above inner product gives rise to a left invariant metric on $SO(3)$. Let E be the ellipsoid on $\mathbb{R}^3 = so(3)$ given by $2H_I = 1$. If we set $x = x_1^2 + x_2^2 + x_3^2$ then formula (21) gives the period τ of the periodic orbits of X_{H_I} on E. We have:

(62)
$$\tau(x) = 4\sqrt{I_1 I_2 I_3} \int_0^{\pi/2} \frac{du}{\sqrt{(I_3 - I_2)(x - I_1) - (I_2 - I_1)(I_3 - x)\sin^2 u}}, \quad x \in (I_2, I_3).$$

For $x \in (I_1, I_2)$ the formula for τ is obtained from the above by permuting the indices 1 and 3.

The derivative of τ is easy to compute. One finds that for $x \in (I_1, I_2)$, $\frac{d\tau}{dx}(x) > 0$ and for $x \in (I_2, I_3)$, $\frac{d\tau}{dx}(x) < 0$. We also get

$$\tau(I_1^+) = \lim_{x \to I_1^+} \tau(x) = 2\pi \sqrt{\frac{I_1 I_2 I_3}{(I_1 - I_2)(I_1 - I_3)}}, \tag{63}$$

$$\tau(I_3^-) = \lim_{x \to I_3^-} \tau(x) = 2\pi \sqrt{\frac{I_1 I_2 I_3}{(I_3 - I_2)(I_3 - I_1)}}, \tag{64}$$

$$\lim_{x \to I_2} \tau(x) = \infty. \tag{65}$$

One can check that $\tau(I_1^+) \geq \tau(I_3^-)$ if and only if $I_2 \leq \frac{I_1+I_3}{2}$.

Consider now the left invariant metric on $SO(3)$ defined by H_I. Let $SO(2)$ be any one-parameter subgroup. Then $SO(2)$ acts on $SO(3)$ from the left by isometries. The quotient, M_{I_1,I_2,I_3} is a 2-sphere, and we endow it with the submersion metric. This corresponds to the classical "Poisson reduction" and M is called *the Poisson sphere* [3]. It follows from a theorem of Lusternik and Schnirelmann [15] and estimates of Klingenberg and Toponogov that any convex metric on S^2 whose Gaussian curvature satisfies $1/\Delta < K < \Delta$, has at least three geometrically different closed geodesics with length in $(2\pi/\sqrt{\Delta}, 2\pi\sqrt{\Delta})$. That this is optimal is shown by a result of Morse:

Given any constant $N > 2\pi$ there exists an $\varepsilon > 0$ such that any prime closed geodesic on an ellipsoid

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 1, \quad a_1 < a_1 < a_3$$

and $\mid 1 - a_i \mid < \varepsilon$, is either a principal ellipse or it has length larger than N.

Using the observations from the previous section we will prove a similar result for the Poisson sphere. It should be noted that the Poisson sphere and the ellipsoid are *not isometric*. In fact their geodesic flows are not topologically conjugate even though there is a homeomorphism that takes orbits into orbits [5, 22].

Theorem 4.10. *Given $N > 2\pi$ there exists an $\varepsilon > 0$ such that any prime closed geodesic on the Poisson sphere M_{I_1,I_2,I_3} with $\mid 1 - I_i \mid < \varepsilon$ has length $> N$, except for three closed geodesics with length close to 2π.*

Proof. According to Corollary 4.8 the Hamiltonian associated with the metric on M_{I_1,I_2,I_3} is collective for the canonical action of $SO(3)$ on $T^*(M)$ and its defining function is $f = \frac{x_1^2}{I_1} + \frac{x_2^2}{I_2} + \frac{x_3^2}{I_3}$. Consider the sphere bundle S in $T^*(M)$. Then the moment map ψ of the $SO(3)$-action on $T^*(M)$ is a submersion from S to E where E is the ellipsoid $\frac{x_1^2}{I_1} + \frac{x_2^2}{I_2} + \frac{x_3^2}{I_3} = 1$. Let us apply the description of collective motion from Section 4. We know that the Hamiltonian flow of f restricted to E has six critical points, 4 heteroclinic connections and closed orbits with period $\tau(x)$ where $x = x_1^2 + x_2^2 + x_3^2$.

The six critical points give rise to geodesics which are orbits of one-parameter subgroups, namely the one-parameter subgroups generated by $(\pm\sqrt{I_1}, 0, 0)$, $(0, \pm\sqrt{I_2}, 0)$ and $(0, 0, \pm\sqrt{I_3})$. Geometrically we only get three different closed geodesics whose length is clearly close to 2π if $\mid 1 - I_i \mid < \varepsilon$. Note that since $\psi : S \to E$ is a submersion, those are the only geodesics which are orbits of one-parameter subgroups.

Now, suppose $x(t)$ is a closed geodesic with length L, different from the ones described above. Then $\phi(x(t))$ is a closed curve in E. Thus $L \geq \tau(x)$ for all $x \in (I_1, I_2)$ and all $x \in (I_2, I_3)$. In other words

$$L \geq \min\{\tau(I_3^-), \tau(I_1^+)\}$$

But from the equations (63) and (64) we see that given N there exists $\varepsilon > 0$ so that if $\mid 1 - I_i \mid < \varepsilon$ then $\min\{\tau(I_3^-), \tau(I_1^+)\} > N$.

□

4.4. **Criterion 1.** Let $\mathfrak{g}$ be a real form of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ and let G be a semi-simple Lie group with Lie algebra $\mathfrak{g}$. Let $(M, \mathcal{P})$ be a Hamiltonian G manifold. Recall that this means there is a Poisson map $\psi : (M, \mathcal{P}) \to (\mathfrak{g}^*, \mathcal{P}_{can})$ such that ψ is G-equivariant.

We will say that the G action on M is full if there exists an $m \in M$ such that $\dim G.m \geq 4$.

Criterion 1. *Let M be a Hamiltonian G space, and assume that the G action is full. Then, $\psi(M)$ contains regular coadjoint orbits*

Proof. Because G's action is Hamiltonian, the actions of G_1 and G_2 are also Hamiltonian. The moment maps are $\psi_i : M \to \mathfrak{g}_i^*$ given by $\psi_i = \pi_i \circ \psi$, where $\pi_i : \mathfrak{g}^* \to \mathfrak{g}_i^*$ is the transpose of the inclusion map. Equivalently, $\psi = \psi_1 \oplus \psi_2$ where $\mathfrak{g}_1^*$ is canonically identified with the subspace of $\mathfrak{g}^*$ that vanishes on $\mathfrak{g}_2$, and vice versa for $\mathfrak{g}_2^*$.

We wish to show that $\psi(M)$ contains a regular element. We will do this by *reductio ad absurdum.* Let $R := \{m \in M \, : \, \dim G.m \geq 4\}$. By hypothesis, $R \neq \emptyset$, and since R contains the regular G orbits, it is open and dense in M. Assume that for all $m \in R$, that $\dim \mathcal{O}_\mu \leq 2$ where $\mu = \psi(m)$. Let us remark that if $\dim \mathcal{O}_\mu < 2$ for all $\mu \in \psi(M)$, then $\dim \mathcal{O}_\mu = 0$ and so $\psi(M) = \{0\}$. This implies that the Hamiltonians induced by G's action are all trivial, hence the G action is trivial. Absurd. Thus, there exists an $m \in R$ such that either $\psi_1(m) \neq 0$ or $\psi_2(m) \neq 0$. Without loss of generality, we may suppose that $\psi_2(m) \neq 0$. Consequently, the set $S := \{m \in R \, : \, \psi_2(m) \neq 0\}$ is open and dense in M.

Let $m \in S$. Then, since $\dim \mathcal{O}_\mu \leq 2$, and $\psi_2(m) \neq 0$, we have that $\psi_1(m) = 0$. This shows that $\psi_1|S = 0$, and hence $\psi_1 \equiv 0$. Therefore G_1 acts trivially on M, which implies that for all $m \in M$, $\dim G.m = \dim G_2.m \leq 3$. Absurd. Therefore, $\psi(M)$ contains regular, 4-dimensional coadjoint orbits.

□

4.5. **Criterion 2.** Suppose that H is a semi-simple Lie group, $\mathfrak{h}$ =Lie(H), and a real form $\mathfrak{g}$ of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ embeds into $\mathfrak{h}$; let $G \leq H$ be a subgroup with Lie algebra $\mathfrak{g}$. Let T^*H be trivialized by the left action of H, so that $T^*H = H \times \mathfrak{h}^*$. The moment map of the right action of H on T^*H in this trivialization is just projection onto the second factor: $\psi_R(h,p) = p$. The moment map of the left action is given by the coadjoint action of H on $\mathfrak{h}^*$: $\psi_L(h,p) = \mathrm{Ad}_h^* p$. Let $K \subset H$ be a closed subgroup, $\mathfrak{k} \overset{i}{\hookrightarrow} \mathfrak{h}$ its Lie algebra and $\pi : \mathfrak{h}^* \to \mathfrak{k}^*$ the transpose of the inclusion map. The moment map of K's left action on H is given by $\psi_{L,K}(h,p) = \pi\psi_L(h,p)$. Let $X = K\backslash H$. Then T^*X is symplectomorphic to $K\backslash\psi_{L,K}^{-1}(0) = K\backslash(H \times \mathfrak{k}^\perp)$. The right action of H commutes with K's left action on T^*H, and so we can define the moment map of H's right action on T^*X. We have

$$\psi_{R,T^*X}(K(id,p)) = i_\perp(p) \tag{66}$$

where $p \in \mathfrak{k}^\perp$ is a representative of the coset $K(id,p) \in T^*X$ and $i_\perp : \mathfrak{k}^\perp \to \mathfrak{h}^*$ is the inclusion map. If we let $\pi_\mathfrak{g} : \mathfrak{h}^* \to \mathfrak{g}^*$ denote the transpose to the inclusion map

$i_{\mathfrak{g}} : \mathfrak{g} \hookrightarrow \mathfrak{h}$, then we have that the moment map of G's right action on T^*X satisfies

$$\psi_{R,T^*X,G}(K(id,p)) = \pi_{\mathfrak{g}} \circ i_{\perp}(p). \tag{67}$$

Because H is semi-simple, we identify $\mathfrak{h}^* \simeq \mathfrak{h}$ via the Cartan-Killing form $(,)$. Then, $\mathfrak{k}^{\perp}$ is the $(,)$-orthogonal complement to $\mathfrak{k}$, and $\pi_{\mathfrak{g}}$ is the orthogonal projection of $\mathfrak{h} = \mathfrak{g} + \mathfrak{g}^{\perp}$ onto $\mathfrak{g}$. Thus, $\operatorname{im} \psi_{R,T^*X,G}$ contains a regular element in $\mathfrak{g}^* \simeq \mathfrak{g}$ iff $\pi_{\mathfrak{g}} \circ i_{\perp}$ contains a regular element iff $\mathfrak{g} \cap \mathfrak{k}^{\perp} \subset \mathfrak{h}$ contains a regular element. This proves:

Criterion 2. *The moment map $\psi : T^*(K\backslash H) \to \mathfrak{g}^*$ hits a regular orbit iff $\mathfrak{g} \cap \mathfrak{k}^{\perp}$ contains a regular element.*

A *subspace* $V \subset \mathfrak{g}$ contains a regular element if $\dim V \geq 4$. Thus, if $\dim \mathfrak{g} \cap \mathfrak{k}^{\perp} \geq 4$, then $\mathfrak{g} \cap \mathfrak{k}^{\perp}$ contains a regular element and Criterion 2 can be applied.

5. Collective geodesic flows

We now show using Criterion 1 and 2 that for all the spaces $K\backslash H$ listed in Theorem C, there exists an embedding of a real form $\mathfrak{g}$ of $sl(2;\mathbb{C}) \oplus sl(2;\mathbb{C})$ into $\mathfrak{h}$ such that the moment map $\psi : T^*(K\backslash H) \to \mathfrak{g}^*$ hits a regular orbit. A combination of this fact with a similar argument to the one we used to prove the corollary after Theorem B and the results in Subsection 4.1 yields Theorem C. Observe that any collective Hamiltonian Poisson commutes with the pull back by the moment map of the Cartan-Killing quadratic form, hence our collective Riemannian metrics Poisson commute with the bi-invariant metric.

Stiefel Manifolds: Let $H = \mathrm{SO}(p+q)$, $K = \mathrm{SO}(q)$; the Stiefel manifold of p-dimensional orthonormal frames in $\mathbb{R}^{p+q}$ is $V_{p+q,p}(\mathbb{R}) = K\backslash H$. For $p \geq 1$ and $q \geq 3$ or $p, q \geq 2$ we can embed $K = \mathrm{SO}(q)$ into the lower right corner and $G = \mathrm{SO}(4)$ into the upper right corner of H so that

$$\mathfrak{g} \cap \mathfrak{k}^{\perp} \supseteq \left[\begin{array}{c|ccc|ccc} 0 & * & * & * & 0 & \cdots & 0 \\ \hline * & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right],$$

and it is clear that $\mathfrak{g} \cap \mathfrak{k}^{\perp}$ contains a regular element of $\mathfrak{g}$. Note that when $p = 1$ and $q = 3$ (the 3-sphere) $\dim \mathfrak{g} \cap \mathfrak{k}^{\perp} = 3$, but still $\mathfrak{g} \cap \mathfrak{k}^{\perp}$ contains a regular element.

BDI: Oriented Grassmannian Manifolds: Let $H = \mathrm{SO}(p+q)$, $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$; the Grassmannian manifold of oriented p-dimensional planes in $\mathbb{R}^{p+q}$ is $G_{p+q,p}(\mathbb{R}) = K\backslash H$. For $p \geq 1$ and $q \geq 3$ or $p, q \geq 2$, we can embed K into the diagonal of H and $G = \mathrm{SO}(4)$ along the diagonal of H so that

$$\mathfrak{g}\cap\mathfrak{k}^{\perp} \supseteq \left[\begin{array}{cc|cccc} \mathrm{so}(p) & & & & & \\ & 0 & * & * & * & \\ \hline & * & 0 & 0 & 0 & \\ & * & 0 & 0 & 0 & \\ & * & 0 & 0 & 0 & \\ & & & & & \mathrm{so}(q) \end{array}\right].$$

It is clear that $\mathfrak{g}\cap\mathfrak{k}^{\perp}$ contains a regular element of $\mathfrak{g}$.

Complex Stiefel Manifolds: Let $H = \mathrm{SU}(p+q)$, $K = \mathrm{SU}(q)$; the Stiefel manifold of p-dimensional orthonormal frames in $\mathbb{C}^{p+q}$ is $V_{p+q,p}(\mathbb{C}) = K\backslash H$. For $p \geq 1$ and $q \geq 3$, or $p, q \geq 2$, we can embed K and $G = \mathrm{SO}(4)$ as in the real case.

AIII: Complex Grassmannian Manifolds: Let $H = \mathrm{SU}(p+q)$, $K = \mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$; the Grassmannian manifold of p-dimensional planes in $\mathbb{C}^{p+q}$ is $G_{p+q,p}(\mathbb{C}) = K\backslash H$. For $p \geq 1$ and $q \geq 3$, or $p, q \geq 2$, we can embed K and $G = \mathrm{SO}(4)$ as in the real case.

AI: $\mathrm{SO}(m)\backslash\mathrm{SU}(m)$: For $m \geq 4$, we can embed $G = \mathrm{SU}(2)\times\mathrm{SU}(2)$ into the upper left corner of $H = \mathrm{SU}(m)$. Then $K\cap G = \mathrm{SO}(2)\times\mathrm{SO}(2)$, so it is clear that $\mathfrak{g}\cap\mathfrak{k}^{\perp}$ contains a generic element.

AII: $\mathrm{Sp}(m)\backslash\mathrm{SU}(2m)$: The group $K = \mathrm{Sp}(m)$ is the maximal compact subgroup of $\mathrm{Sp})m;\mathbb{C})$. By embedding $G = \mathrm{SO}(4)$ in the upper left corner of $\mathrm{SU}(2m)$, $m \geq 2$, we have $G\cap K = \mathrm{SO}(2)\times\mathrm{SO}(2)$. This implies that $\mathfrak{g}\cap\mathfrak{k}^{\perp}$ contains a regular element.

CI: $\mathrm{U}(m)\backslash\mathrm{Sp}(m)$: Embed the group $G = \mathrm{Sp}(1)\oplus\mathrm{Sp}(1) \simeq \mathrm{SU}(2)\oplus\mathrm{SU}(2)$ into $H = \mathrm{Sp}(m)$ along the diagonal for $m \geq 2$. Then $G\cap K = U(1)\oplus U(1)$.

CII: $\mathrm{Sp}(p)\oplus\mathrm{Sp}(q)\backslash\mathrm{Sp}(p+q)$: The inclusion $\mathbb{R}\hookrightarrow\mathbb{C}\hookrightarrow\mathbb{H}$ induces the obvious inclusion $O(n;\mathbb{R})\hookrightarrow O(n;\mathbb{C})\hookrightarrow O(n;\mathbb{H})$ which allows us to embed $G = O(4)$ into $\mathrm{Sp}(n) = O(n;\mathbb{H})$ for $n \geq 4$. The condition on $\mathfrak{g}\cap\mathfrak{k}^{\perp}$ is satisfied for $p = 1$, $q \geq 3$ and $p, q \geq 2$. For $p = 1$ and $q = 2$ we have to observe that $\mathrm{Sp}(1)\oplus\mathrm{Sp}(2)\backslash\mathrm{Sp}(3) = \mathbb{H}P^2$. The group $G = \mathrm{Sp}(1)\oplus\mathrm{Sp}(1)$ acts on $\mathbb{H}P^2$ by $(g,h)*[x:y:z] := [gx:hy:z]$ where $\mathrm{Sp}(1)$ is identified with the group of unit quaternions. Since the G stabilizer of $[1:1:1]$ is trivial, the action of G is full. Hence, we can apply Criterion 1.

DIII: $U(m)\backslash\mathrm{SO}(2m)$: We embed $G = \mathrm{SO}(3)\oplus\mathrm{SO}(3)$ along the diagonal of $\mathrm{SO}(6)$ in the obvious way. Then $K\cap G = \Delta(G) = \{(g,g) \ : \ g\in G\}$, and so $\mathfrak{g}\cap\mathfrak{k}^{\perp}$ contains a regular element for $m \geq 2$.

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