COMPLEX ANALYSIS EXAMPLES 1

G.P. Paternain Lent 2018

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Let $T\colon \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}^2 = \mathbb{C}$ be a real linear map. Show that there exist unique complex numbers A, B such that for every $z \in \mathbb{C}$, $T(z) = Az + B\bar{z}$. Show that T is complex differentiable if and only if $B = 0$.

2. (i) Let $f : D \to \mathbb{C}$ be an holomorphic function defined on a domain D. Show that f is constant if any one of its real part, imaginary part, modulus or argument is constant.

(ii) Find all holomorphic functions on $\mathbb{C}$ of the form $f(x+iy) = u(x) + iv(y)$ where u and v are both real valued.

(iii) Find all holomorphic functions on $\mathbb{C}$ with real part $x^3 - 3xy^2$.

3. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$, and

$$f(z) = \frac{(1+i)x^3 - (1-i)y^3}{x^2+y^2} \quad \text{for } z = x + iy \neq 0.$$

Show that f satisfies the Cauchy-Riemann equations at 0 but is not differentiable there.

4. (i) Define the differential operators $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$. Prove that a C^1 function f is holomorphic iff $\partial f/\partial \bar{z} = 0$. Show that

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the usual Laplacian in $\mathbb{R}^2$.

(ii) Let $f : U \to V$ be holomorphic and let $g : V \to \mathbb{C}$ be harmonic. Show that the composition $g \circ f$ is harmonic.

5. (i) Denote by Log the principal branch of the logarithm. If $z \in \mathbb{C}$, show that $n\,\mathrm{Log}(1+z/n)$ is defined if n is sufficiently large, and that it tends to z as n tends to ∞. Deduce that for any $z \in \mathbb{C}$,

$$\lim_{n\to\infty}\left(1 + \frac{z}{n}\right)^n = e^z.$$

(ii) Defining $z^\alpha = \exp(\alpha\,\mathrm{Log}\,z)$, where Log is the principal branch of the logarithm and $z \notin \mathbb{R}_{\leq 0}$, show that $\frac{d}{dz}(z^\alpha) = \alpha z^{\alpha-1}$. Does $(zw)^\alpha = z^\alpha w^\alpha$ always hold?

6. Prove that each of the following series converges uniformly on compact (i.e. closed and bounded) subsets of the given domains in $\mathbb{C}$:

$$(a)\ \sum_{n=1}^{\infty} \sqrt{n}e^{-nz} \quad \text{on } \{\, z : \ 0 < \mathrm{Re}(z)\,\}; \qquad (b)\ \sum_{n=1}^{\infty} \frac{2^n}{z^n + z^{-n}} \quad \text{on } \left\{ z : \ |z| < \frac{1}{2} \right\}.$$

7. Find conformal equivalences between the following pairs of domains:

(i) the sector $\{z \in \mathbb{C} : \ -\pi/4 < \arg(z) < \pi/4\}$ and the open unit disc $D(0,1)$;

(ii) the lune $\{z \in \mathbb{C} : |z-1| < \sqrt{2} \text{ and } |z+1| < \sqrt{2}\}$ and $D(0,1)$;

(iii) the strip $S = \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < 1\}$ and the quadrant $Q = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$.

By considering a suitable bounded solution of Laplace's equation $u_{xx} + u_{yy} = 0$ on S, find a non-constant harmonic function on Q which is constant on each of the two boundaries of the quadrant (it need not be continuous at the origin).

8. (i) Show that the most general Möbius transformation which maps the unit disk onto itself has the form $z \mapsto \lambda \dfrac{z-a}{\bar{a}z-1}$, with $|a| < 1$ and $|\lambda| = 1$. [*Hint: first show that these maps form a group.*]

(ii) Find a Möbius transformation taking the region between the circles $\{|z| = 1\}$ and $\{|z-1| = 5/2\}$ to an annulus $\{1 < |z| < R\}$. [*Hint: a circle can be described by an equation of the shape* $|z-a|/|z-b| = \ell$.]

(iii) Find a conformal map from an infinite strip onto an annulus. Can such a map ever be a Möbius transformation?

9. Let $f : U \to \mathbb{C}$ be a holomorphic function where U is an open set (you may assume f is also C^1). Let $z_0 \in U$ be a point such that $f'(z_0) \neq 0$. Use the inverse function theorem from Analysis II to show that f is a conformal equivalence locally around z_0.

10. Calculate $\int_\gamma z \sin z \, dz$ when γ is the straight line joining 0 to i.

11. Show that the following functions do not have antiderivatives (i.e. functions of which they are the derivatives) on the domains indicated:

$$(a) \quad \frac{1}{z} - \frac{1}{z-1} \quad (0 < |z| < 1)\,; \qquad (b) \quad \frac{z}{1+z^2} \quad (1 < |z| < \infty).$$