

## ANALYSIS I EXAMPLES 1

**G.P. Paternain Lent 2010**

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at [g.p.paternain@dpmmms.cam.ac.uk](mailto:g.p.paternain@dpmmms.cam.ac.uk).

1. Prove that if  $a_n \rightarrow a$  and  $b_n \rightarrow b$  then  $a_n + b_n \rightarrow a + b$ .
2. Sketch the graphs of  $y = x$  and  $y = (x^4 + 1)/3$ , and thereby illustrate the behaviour of the real sequence  $a_n$  where  $a_{n+1} = (a_n^4 + 1)/3$ . For which of the three starting cases  $a_1 = 0$ ,  $a_1 = 1$  and  $a_1 = 2$  does the sequence converge? Now prove your assertion.
3. Let  $a_1 > b_1 > 0$  and let  $a_{n+1} = (a_n + b_n)/2$ ,  $b_{n+1} = 2a_n b_n / (a_n + b_n)$  for  $n \geq 1$ . Show that  $a_n > a_{n+1} > b_{n+1} > b_n$  and deduce that the two sequences converge to a common limit. What limit?
4. The real sequence  $a_n$  is bounded but does not converge. Prove that it has two convergent subsequences with different limits.
5. Investigate the convergence of the following series. For those expressions containing the complex number  $z$ , find those  $z$  for which convergence occurs.

$$\sum_n \frac{\sin n}{n^2} \quad \sum_n \frac{n^2 z^n}{5^n} \quad \sum_n \frac{(-1)^n}{4 + \sqrt{n}} \quad \sum_n \frac{z^n (1-z)}{n}$$

6. Show that  $\sum 1/(n \log^\alpha n)$  converges if  $\alpha > 1$  and diverges otherwise. Does  $\sum 1/(n \log n \log \log n)$  converge?

7. Consider the two series  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$  and  $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$ , having the same terms but taken in a different order. Let  $s_n$  and  $t_n$  be the corresponding partial sums to  $n$  terms. Show that  $s_{2n} = H_{2n} - H_n$  and  $t_{3n} = H_{4n} - \frac{1}{2}H_{2n} - \frac{1}{2}H_n$ , where  $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n}$ . Show that  $s_n$  converges to a limit  $s$  and that  $t_n$  converges to  $3s/2$ .

8. Suppose that  $\sum a_n$  diverges and  $a_n > 0$ . Show that there exist  $b_n$  with  $b_n/a_n \rightarrow 0$  and  $\sum b_n$  divergent.

9. (Abel's test.) Let  $a_n$  and  $b_n$  be two sequences and let  $S_n = \sum_{j=1}^n a_j$  and  $S_0 = 0$ . Show that for any  $1 \leq m \leq n$  we have:

$$\sum_{j=m}^n a_j b_j = S_n b_n - S_{m-1} b_m + \sum_{j=m}^{n-1} S_j (b_j - b_{j+1}).$$

Suppose now that  $b_n$  is a decreasing sequence of positive terms tending to zero. Moreover, suppose that  $S_n$  is a bounded sequence. Prove that  $\sum_{j=1}^\infty a_j b_j$  converges. Deduce the alternating series test.

Does the series  $\sum_{n=1}^\infty \frac{\cos(n)}{n}$  converge or diverge?

10. For  $n \geq 1$ , let

$$a_n = \frac{1}{\sqrt{n}} + \frac{(-1)^{n-1}}{n}.$$

Show that each  $a_n$  is positive and that  $\lim a_n = 0$ . Show also that  $\sum_{n=1}^\infty (-1)^{n-1} a_n$  diverges. [This shows that, in the alternating series test, it is essential that the moduli of the terms decrease as  $n$  increases.]

**11.** Let  $z \in \mathbb{C}$ . Show that the series

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \frac{z^4}{1-z^8} + \frac{z^8}{1-z^{16}} + \dots$$

converges to  $z/(1-z)$  if  $|z| < 1$ , converges to  $1/(1-z)$  if  $|z| > 1$ , and diverges if  $|z| = 1$ .

**12.** Prove that every real sequence has a monotonic subsequence. Deduce the Bolzano-Weierstrass theorem.

**13.** Can we write the open interval  $(0,1)$  as a disjoint union of closed intervals of positive length?

**14.** In lectures (or problem 12) you learned a proof of the Bolzano-Weierstrass theorem based on the fundamental axiom (every increasing sequence bounded above converges). Show that the Bolzano-Weierstrass theorem is in fact equivalent to the fundamental axiom, that is, give a proof of the fundamental axiom assuming Bolzano-Weierstrass.