Sparse superconcentrators

David Ellis

We have just seen that for any $d \geq 3$, there exist arbitarily large d-regular graphs with edge-expansion ratio at least $c_d d$ (where we can take $c_d = 0.18$ for all d). Such graphs are 'sparse', having only $dn/2$ edges, and yet have vertex-expansion ratio at least c_d — i.e., they are good vertex-expanders. As we will see later, there are now many explicit constuctions of arbitrarily large vertex-expanders of bounded degree, among them the Ramanujan graphs of Lubotzsky-Phillips-Sarnak / Margulis. Our aim in this section is to show how large, bounded-degree vertex-expanders can be used to construct sparse *superconcentrators.*

Definition. *An* n-superconcentrator *is a directed graph* D *containing two disjoint sets of vertices,* X *(the 'senders') and* Y *(the 'receivers'), with* $|X| = |Y| = n$*, such that for any* $k \leq n$*, and for any two subsets* $S \subset X$ *and* $T \subset Y$ *with* $|S| = |T| = k$*, there exist* k *vertex-disjoint, directed paths in* D *from* S *to* T*.*

As the reader may imagine, superconcentrators are important in communication theory and theoretical computer science. It is natural to ask, how few edges can an n-superconcentrator have? We write $\phi(n)$ for this minimum. Valiant was the first to consider this question; at first, he conjectured that $\phi(n)$ must be superlinear in n. However, it turns out that $\phi(n)$ is linear in n. We will now give an iterative construction of n-superconcentrators with at most Cn edges, for some absolute constant C, due to Gabber and Galil [2]. First, however, we will want a way of constructing arbitrarily large *bipartite* expanders.

Definition. *A bipartite graph* B *with vertex-classes* X *and* Y *is said to be a* β-right-vertex-expander *if*

$$|\Gamma_B(S)| \geq (1+\beta)|S| \quad \forall S \subset X \text{ with } |S| \leq |X|/2.$$

It is said to be a β-left-vertex-expander *if*

$$|\Gamma_B(T)| \geq (1+\beta)|T| \quad \forall T \subset Y \text{ with } |T| \leq |Y|/2.$$

There is an easy way of constructing a β-right-vertex expander with both vertex-classes of size n from an n-vertex graph with vertex-expansion ratio at least β. Suppose $G = (V, E)$ is an n-vertex graph with vertex-expansion ratio at least β. Form a bipartite graph $B = B_G$ with vertex-classes consisting of two disjoint copies of V, $V \times \{1\}$ and $V \times \{2\}$, where we join $(v, 1)$ to $(w, 2)$ whenever $v = w$ or $vw \in E(G)$. Clearly, B_G is a β-right-vertex-expander with both vertex-classes of size n, and with $e(G) + n$ edges. (It is also a β-left-vertex-expander, but we will not need this.)

Given an n-superconcentrator D, and an n-vertex, d-regular graph with vertex-expansion ratio at least β, we will give a construction of a $\lfloor (1+\alpha)n \rfloor$-superconcentrator $\tilde{D}$ (for some $\alpha > 0$ depending upon β) which only has at most $K(\alpha, d)n$ more edges than D, for some constant $K(\alpha, d)$ depending only upon α and d. (This will suffice to describe an interative construction of n-superconcentrators with at most Cn edges, for some absolute constant $C > 0$, for every $n \in \mathbb{N}$.)

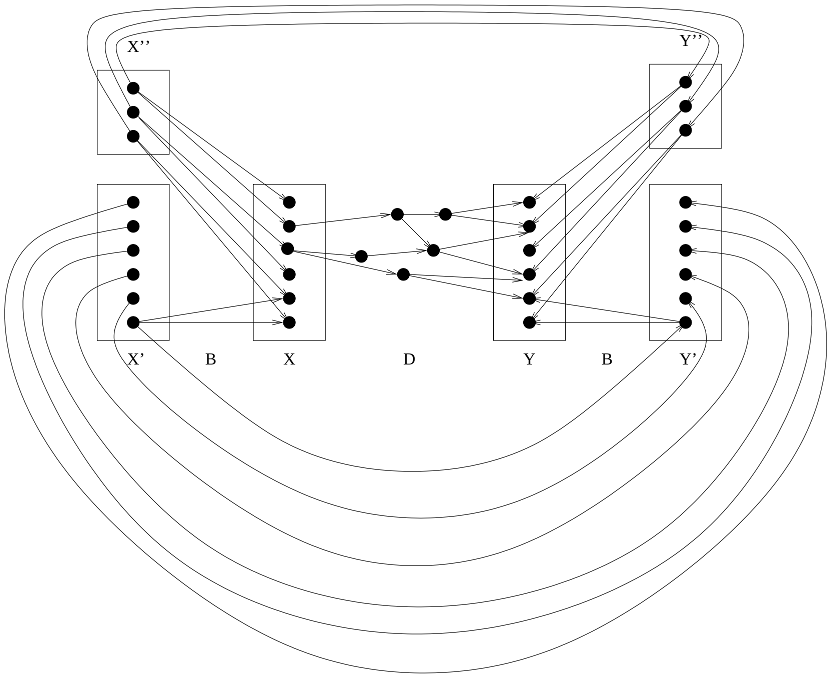
Suppose we have at our disposal an n-vertex, d-regular graph $G = (V, E)$ with vertex-expansion ratio at least β. Start with D; let X and Y be the senders and receivers of D, respectively. Choose $\alpha \leq \beta/2$ maximal such that $1/\alpha \in \mathbb{N}$; let $s = 1/\alpha$.

Add a set of n extra vertices, X', to the left of X. Identify X' with $V \times \{1\}$ and X with $V \times \{2\}$, and draw between X' and X the edges of $B = B_G$, directed from X' to X. Now add an extra set of n vertices, Y', to the right of Y. Identify Y' with $V \times \{1\}$, and Y with $V \times \{2\}$, and draw between Y' and Y the edges of B_G, this time directed from Y to Y'.

Now place a set X'' of $\lfloor \alpha n \rfloor$ extra vertices above X', and another set of $\lfloor \alpha n \rfloor$ extra vertices above Y'. Partition the first $s\lfloor \alpha n \rfloor$ vertices of X into $\lfloor \alpha n \rfloor$ disjoint classes of size s, and draw directed edges from the ith vertex of X'' to each of the s vertices in the ith class of X, for every i. Similarly, partition the first $s\lfloor \alpha n \rfloor$ vertices of Y into αn disjoint classes of size s, and draw directed edges from the ith vertex of Y'' to each of the s vertices in the ith class of Y, for every i.

Let $\tilde{X} = X' \cup X''$, and let $\tilde{Y} = Y' \cup Y''$; we have $|\tilde{X}| = |\tilde{Y}| = \lfloor (1 + \alpha)n \rfloor$. Let L denote the bipartite directed graph we have drawn between $\tilde{X}$ and X, and let R denote the one we have drawn between $\tilde{Y}$ and Y. As we will see (Lemma 1), L has the crucial property that for any $S \subset \tilde{X}$ with $|S| \leq |\tilde{X}|/2$, $\Gamma_L(S)| \geq |S|$, i.e. the right-neighbourhoods of (small) sets are as large as the sets themselves, even though L goes from a $\lfloor (1 + \alpha)n \rfloor$-set to an n-set. We call a directed bipartite graph with this property an $(1 + \alpha, n)$-*collapser*. Of course, R is also an $(1 + \alpha, n)$-collapser.

Finally, we choose any bijection $\psi : \tilde{X} \to \tilde{Y}$, and add in the $\lfloor (1 + \alpha)n \rfloor$ edges $\{(x, \psi(x)) : x \in \tilde{X}\}$, forming a directed matching M from $\tilde{X}$ to $\tilde{Y}$. Let $\tilde{D} = D \cup L \cup R \cup M$ be the resulting directed graph:



Observe that

$$e(\tilde{D}) \leq e(D) + 2(d+1)n + 2n + (1+\alpha)n.$$

We will now show that $\tilde{D}$ is a $\lfloor (1+\alpha)n \rfloor$-superconcentrator with senders $\tilde{X} = X' \cup X''$, and receivers $\tilde{Y} = Y' \cup Y''$.

First, we need the following

Lemma 1. *For any $S \subset \tilde{X}$ with $|S| \leq (1+\alpha)n/2$, we have $|\Gamma_L(S)| \geq |S|$.*

Proof. Write $S = S' \cup S''$, where $S' = S \cap X'$ and $S'' = S \cap X''$. If $|S'| \geq n/2$, then

$$|\Gamma_B(S')| \geq (1+\beta)|S'| \geq (1+\beta)n/2 \geq (1+2\alpha)n/2 \geq |S|,$$

so $|\Gamma_L(S)| \geq |S|$, and we are done. If $|S''| \geq \alpha |S|$, then

$$|\Gamma_L(S'')| = (1/\alpha)|S''| \geq |S|,$$

so $|\Gamma_L(S)| \geq |S|$, and we are done. Hence, we may assume that $|S'| \leq n/2$ and $|S''| \leq \alpha |S|$. But then

$$|\Gamma_B(S')| \geq (1+\beta)|S'| \geq (1+\beta)(1-\alpha)|S| \geq |S|,$$

so $\Gamma_L(S)| \geq |S|$ in this case also, completing the proof. □

It follows from Hall's theorem that for any $S \subset \tilde{X}$ with $|S| \leq (1+\alpha)n/2$, there exists a matching in L from S to a set $\bar{S} \subset X$. By exactly the same argument, for any $T \subset \tilde{Y}$ with $|T| \leq (1+\alpha)n/2$, there exists a matching in R from T to a set $\bar{T} \subset Y$. Hence, for any $k \leq (1+\alpha)n/2$, and any two sets $S \subset \tilde{X}$ and $T \subset \tilde{Y}$ with $|S| = |T| = k$, we can find k vertex-disjoint directed paths from S to T in $\tilde{D}$: match S to $\bar{S}$ using edges of L, match T to $\bar{T}$ using edges of R, and then choose k vertex-disjoint directed paths in D from $\bar{S}$ to $\bar{T}$.

We now reduce the case $k \geq (1+\alpha)n/2$ to the case $k \leq (1+\alpha)n/2$, using the edges of the matching M. Let $S \subset \tilde{X}$ and $T \subset \tilde{Y}$ with $|S| = |T| = k \geq (1+\alpha)n/2$. Let $l = |\psi(S) \cap T|$; note that $l \geq k - \lfloor (1+\alpha)n/2 \rfloor$. Use l edges of the matching M to connect $S \cap \psi^{-1}(T)$ to $\psi(S) \cap T$. Let S^* and T^* be the remaining vertices of S and T; then $|S^*| = |T^*| \leq (1+\alpha)n/2$, so by our argument above, there exist vertex-disjoint paths from S^* to T^* in $L \cup D \cup R$. Combining these with the l edges of the matching M yields k vertex-disjoint paths in $\tilde{D}$ from X to Y.

It follows that $\tilde{D}$ is a $\lfloor (1+\alpha)n/2 \rfloor$-superconcentrator with

$$e(\tilde{D}) \leq e(D) + 2(d+1)n + 2n + (1+\alpha)n.$$

Hence,

$$\phi(\lfloor (1+\alpha)n \rfloor) \leq \phi(n) + 2(d+1)n + 2n + (1+\alpha)n.$$

Let $n_0 \in \mathbb{N}$ be fixed; for each $n \leq n_0$, a complete bipartite graph $K_{n,n}$ gives us an n-superconcentrator with at most $n_0 n$ edges, so $\phi(n) \leq n_0 n$. For $n \geq n_0$, we may apply the argument above (with any fixed $d \geq 3$, and with $\beta = 0.18$) to give an iterative construction of n-superconcentrators with at most Cn edges, where C is an absolute constant. Observe that the iterative construction has at most $O(\log n)$ stages.

The search for n-superconcentrators with at most Cn edges, with C as small as possible, is ongoing, and has motivated quite a few advances in the study of expander graphs. Using a probabilistic argument, Schöning [3] recently proved that there exist n-superconcentrators with at most $28(1+o(1))n$ edges. The best explicit construction, with $44(1+o(1))n$ edges, is due to Alon and Capalbo [1], and uses the Ramanujan graphs of Lubotzsky, Phillips and Sarnak.

References

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[2] Gabber, O., Galil, Z., Explicit constructions of linear size superconcentrators, *FOCS*, pp. 364-370, 20th Annual Symposium on Foundations of Computer Science (FOCS 1979), 1979.

[3] Schöning, U., Smaller superconcentrators of density 28, *Information processing letters*, Volume 98, Issue 4 (2006), pp. 127-129.